STOCHASTIC BEHAVIOR OF SPOT AND FUTURE COMMODITY PRICES: NUMERICAL METHODS APPROACH

By

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CERTIFICATE

This is to certify that the thesis on **"STOCHASTIC BEHAVIOR OF SPOT AND FUTURE COMMODITY PRICES: NUMERICAL METHODS APPROACH"** by **R K PAVAN KUMAR PANNALA** in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy (Science) is an original work carried out by him under our joint supervision and guidance.

It is certified that the work has not been submitted anywhere else for the award of any other diploma or degree of this or any other University.

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I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text.

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ABSTRACT

Mathematical Models in Finance have been most active areas of research since their inception in 1973 as Black-Scholes model. Various mathematical models in the form of linear partial differential equations, nonlinear partial differential equations, partial-integro differential equations and fractional order partial differential equations were developed in the financial market (commodity market as well as securities market) arena. In commodity market most of the mathematical models are in the form of linear partial differential equations. However, models of all varieties as mentioned above can be found in securities market.

The models in financial market as categorized above were solved effectively using the discretization techniques such as finite difference method (FDM), operator splitting method, alternating directions implicit (ADI) method, higher order compact (HOC) method, front-fixing method, exponential time integration (ETI) method, hybrid finite difference method, penalty method etc.

Apart from the above mentioned discretization techniques, it is found in the literature that the non-discretization techniques such as Adomian decomposition method (ADM), variational iteration method (VIM), homotopy perturbation method (HPM) and homotopy analysis method (HAM) are also very effective in solving linear partial differential equation (Black-Scholes equation) and obtained the solution in the form of an approximate polynomial. At the same time analytical solution techniques such as first integral method (FIM), tanh-coth and sine-cosine methods in various fields of engineering and science also offer effective and exact solution for nonlinear partial differential equations.

The present study focusses on the solution of nonlinear Black-Scholes partial differential equation of securities market and the solution of linear partial differential equations of commodity market. This work is devoted to study the efficacy of the mentioned analytical methods on the nonlinear Black-Scholes equation and the non-discretization techniques on the linear partial differential equations with variable coefficients proposed by Schwartz and termed as one factor, two factor and three factor commodity price models.

This study is divided into nine Chapters and an Appendix. The first chapter is devoted for introduction to understand the financial market and need of mathematical modeling in financial mathematics. The second chapter is devoted to the literature survey of financial mathematical models and its solution techniques. Chapter 3 presents the description of the non-discretization and analytical solution techniques which are to be used in the study. Chapter 4 is devoted to the effectiveness of the analytical solution techniques on nonlinear Black-Scholes equation. Chapters 5, 6, and 7 devote to testing the efficiency of non-discretization solution techniques ADM, VIM, HPM and HAM on one factor, two factor and three factor Commodity Price models respectively. Chapter 8 is devoted to the conclusions. Finally, Chapter 9 is devoted to the future scope based on this study. The Appendix contains the solutions of the commodity price models and its convergence results obtained using ADM, VIM and HAM.

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NOMENCLATURE

α	Log run mean price of the convenience yield
α_1	Speed rate of variance
α_2	Mean value of variance
α_b	Measure of the price slippage impact of a trade
γ	Stochastic volatility of variance process
γ_0	Risk aversion factor
ϑ	Variance process
λ_j	Jump intensity
$\tilde{\lambda}$	Market price of risk
μ	Drift rate
μ_0	Proportional transaction cost
ν	Levy measure
ξ_0	Constant
ρ_1	Correlation between spot price and convenience yield of a commodity
ρ_2	Correlation between convenience yield and risk free interest rate of a commodity
ρ_3	Correlation between risk free interest rate and spot price of commodity
ρ_l	Measure of liquidity of the market
$\bar{\rho}$	Correlation between stock price and xvariance process
σ	Volatility

σ_1	Volatility of spot price of a commodity
σ_2	Volatility of convenience yield
σ_3	Volatility of the risk free interest rate
σ_c	Time independent volatility
σ_n^2	Non-linear adjusted volatility
φ_0	Volatility correction function
a	Speed adjustment of interest rate
a_s	Arithmetic average of stock price
f	Price of Option
f_0	Size distribution function of finite jump process
g_b	Minimum guaranteed benefit
k	Speed adjustment of commodity spot prices
m^*	Risk adjusted mean short rate of interest rate
q_d	Amount of dividend
r	Risk free interest rate (constant)
s_j	Jump intensity
t	Time
u	Future price of a commodity
x	Spot price of a commodity
y	Convenience yield of a commodity
z	Instantaneous risk free interest rate

$D(t)$	Amount of contribution to the portfolio at time t
K	Strike/exercise price
N_0	Number of options to be sold
$P(t)$	Contract at time t
S	Stock price
T	Maturity date/time
T^- and T^+	Instants immediately before and after the deemed contribution respectively
0^- and 0^+	The time in equilibrium state and non-equilibrium state respectively

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CHAPTER-1

INTRODUCTION

1.1 FINANCIAL MARKETS

Financial market is mainly categorized into two types such as Securities Market and Commodity Market. The main dissimilarity among these two markets is the goods operated. On commodities markets, futures contracts for real commodities are bought and sold, while on the securities market, financier's trade shares of stock in firms. However, the trading of stocks and commodities on these markets are similar. Mostly trading happens on the physical exchanges, such as the New York Stock Exchange (NYSE) and the Chicago Mercantile Exchange (CME). Though, ample trading also happens off the exchange. Stocks in the securities market, sold-off an exchange are said to be sold "over the counter." Commodities sold-off a controlled exchange are said to be sold on the "spot market."

The securities and commodity markets are linked in many ways. In concept, the securities market increases and decreases based on the reported earnings and projected earnings of the companies with stock trading on an index. When business decelerates, or the cost of producing products increases, earnings will fall and so will stock prices. At times, when the commodity markets are rallying, the rise in prices of those commodities means corporate earnings of users of those commodities will decline. Sometimes interference by central banks makes a situation where both commodity markets and securities markets rally or drop.

1.1.1 SPOT & FUTURE PRICES

Trading in financial products has been an important component of world economy. Financial contracts in ‘Spot’ and ‘Future’ trading for both financial securities and commodities have dominated in the international markets. In simple terms, ‘Spot price’ refers to prices at which immediate delivery/receipt of a product is made. ‘Future price’ refers to prices at which delivery/receipt of a product is made on a future date. Financial securities include commercial bills, treasury bills, shares, bonds and so on. Commodities include metals such as gold, silver, copper; crude oil, agricultural products, and so on.

1.1.2 FINANCIAL SECURITIES & COMMODITIES

The trading of these financial securities and commodities happen both at national and international levels involving huge amount of money between the sellers and buyers with the assistance of registered brokers.

Contracts that started on mere gut feel of the participants in the early stages has slowly transformed into trade agreements based on logical analysis for the market conditions and demand and supply factors. Factors such increasing number of multi-national participants, multi-variant products, and high risk of financial manipulations, coupled with policies of different governments across the world mandated for a robust analysis of the price movements of the securities and commodities traded.

1.1.3 STOCHASTIC STUDY – MATHEMATICAL MODELING

The term Stochastic, as given in Oxford Dictionary “characterized by a sequence of random variables” and refers to process. Stochastic study aims to understand the influence in variations of different variables that affect the prices of a product.
The financial industry has both stimulated and benefited from advances in various disciplines of mathematical sciences like probability, differential equations,

Optimization, statistics and numerical analysis. The study of financial systems is centered on the study of the behavior of economic agents in allocating and deploying their resources, both spatially and across time, in an uncertain environment. Time and uncertainty are central elements which influence the financial behavior.

During 1970-1980, a significant research was commenced in development of new models and refinement of existing mathematical models (Black & Scholes [24], Brennan & Schwartz [26]-[28]). By this time the financial databases to support these models were extremely larger and the feasibility of implementing these models was also much greater. The developments in computing and telecommunication technologies made possible the formation of new financial markets and the same technologies made feasible the numerical solution of complex models of multivariate partial differential equations. During this period many mathematicians got attracted to the financial services industry by high salaries and challenging problems.

The advent of mathematical modeling, during the 20^{th} Century, for the financial markets has brought a shift in the paradigm and made the contracts of spot, future, and forwards more reliable. Numerous models were proposed and developed to support contracts in the financial market arena (Andersen & Andersen [12], Company & et al [30], Cont & Voltchkova [31], Esekon [41], Jensen & et al. [75], Kumar & et al. [79], Oosterlee [100], Rodrigoa [108], Schwartz [111], Tangman & et al. [119], Toivanen [122], Vencer [125], Yun [136], [137]). However, all of these evolved in the developed western economies and were dominated the features of those economies.

Economic reforms taken up by the Indian government during 1990’s have contributed for development of their financial systems. With the development of economy, financial markets have also advanced and demands for robust analysis for market trends gained prominence. Models proposed by western researchers are being applied with necessary modifications.

Financial mathematics, also termed as financial engineering, mathematical finance, and computational finance, is the application of mathematical methods to the solution of problems in finance. It imbibes tools from applied mathematics, computer science, statistics, and economic theory for financial models. Investment banks, commercial banks, hedge funds, insurance companies, corporate treasuries, regulatory agencies and commodity traders apply these methods of financial mathematics to problems such as derivative securities valuation, portfolio structuring, risk management, scenario simulation and commodity prices. Quantitative analysis has brought efficiency and rigor to financial markets and to the investment process and is becoming increasingly significant in regulatory concerns. With the pace of financial innovation, the need for highly qualified people with specific training in financial mathematics intensifies.

Finance, offspring of economics, concerns itself with the valuation of assets and financial instruments as well as the apportionment of resources. Centuries of history and experience have resulted in fundamental theories on the way economies function and the way we value assets. Mathematics acts as a suitable tool because it allows theoreticians to model the relationships between variables and represent randomness in a manner that can lead to useful analysis. Mathematics, then, becomes a useful reserve from which researchers can draw to solve problems, provide insights and make the intractable model tractable.

Mathematical finance gathers from the disciplines such as probability theory, statistics, scientific computing and partial differential equations to provide models and derive relationships between fundamental variables like asset prices, market movements, interest rates and convenience yield. These mathematical tools allow us to infer conclusions that can be otherwise difficult to find or not immediately obvious from human intuition.

Support of modern computational techniques help in storage of vast quantities of data and model numerous variables simultaneously, leading to the ability to model quite large and complicated systems. So, it may be inferred that techniques of scientific computing, such as numerical computations, Monte Carlo simulation, and optimization are an important part of financial mathematics.

1.2 BLACK-SCHOLES MODEL

A large part of any science is the ability to develop testable hypotheses based on a fundamental understanding of the objects of study and prove or contradict the hypotheses through repeatable studies. In this light, mathematics is a language to represent theories and provides tools for testing their validity. For instance, in the theory of option pricing due to Black, Scholes and Merton, a model [24], for the movement of stock prices is posited, and in conjunction with basic theory which states that a riskless investment will result in risk-free rate of return, the researchers reasoned that a value can be assigned to an option that is independent of the expected future value of the stock.

This theory, for which Scholes and Merton were awarded the Nobel Prize, is a classic illustration of the interaction between math and financial theory, which ultimately led to a surprising insight into the nature of option prices. The mathematical contribution was the basic stochastic model (Geometric Brownian motion) for movement of stock prices and the partial differential equation (PDE) and its solution providing the relationship between the option's value and other market variables. Their analysis also helped in providing a completely specified strategy for managing option investment which permits practical testing of the model's consequences. This theory, which would not have been possible without the fundamental participation of mathematics, today plays a significant role in a trillion dollar industry.

Over the past three decades the deficiencies of the Black-Scholes model have become progressively clear, with some academic observers continually ringing the “death knell” of the formula as its weaknesses become more obvious and it can lead to substantial discrepancies between actual market prices and prices calculated using the model. These discrepancies between market and theoretical prices are obvious in the observation of different implied volatilities as per the exercise prices (smile or skew) and maturities (term structure). Therefore, despite their popularity and wide spread use, the model is built on some non-real life assumptions about the market. One problem with the Black-Scholes analysis, however, is that the mathematical skills that are required in the derivation and solutions of the model are fairly advanced and probably unfamiliar to many economists.

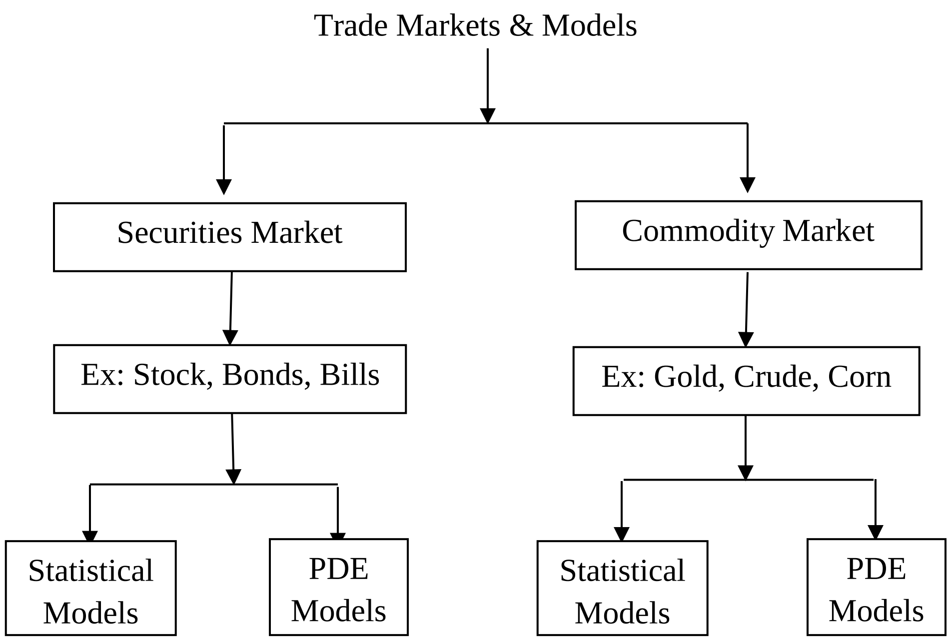

Fig. 1 represents the classification of Financial Markets and Models

Critics of Black-Sholes Model, while contributing on the deficiencies, have proposed further studies in extending the models and also different solution techniques of these models to address numerous financial products. In real world, both the enhanced models and their techniques of tracing suitable solution are significant due to the dynamics of market. They help in prediction of price movement of the financial products with highest precision. These models, often times, are in the form of statistical models and PDE models. However, tracing the accurate solution for these equations is a herculean task which went arrived would help in the development of new mathematical techniques. Statistical models include STARX (McMillan [91]), TAR and STAR (Valtteri [124]), ARIMA (Khan & et al. [78]), and GARCH (Ramirez & Fadiga [106]), etc. PDE models include linear, nonlinear partial differential equations (NPDE) and partial-integro differential equations (PIDE).

Esekon, J. E. [41], developed a nonlinear Black-Scholes model for hedging of derivatives in illiquid markets, and has obtained analytical solution using transformation of variables.

1.3 MATHEMATICAL TECHNIQUES

Arriving at analytical solution is phenomenal assignment since most equations do not have exact methodology for application. However, there are some special methods such as First integral method (FIM) proposed by Feng [49], Tanh-Coth method proposed by Wazwaz [127] and Sine-Cosine method proposed by Alquran & Al-Khaled [11] to get the exact solution of nonlinear PDEs. But, these methods have their own limitations for their applicability such as involving variable coefficients, finding the parameters in Tanh-Coth method and Sine-Cosine methods.

Researchers attempted to develop techniques that could arrive at approximate solution with desired accuracy. These approximate solution techniques have been classified into two categories like discretization and non-discretization techniques.

1.3.1 DISCRETIZATION TECHNIQUES

These include, Finite difference method (FDM) (Brennan & Schwartz [26]-[28]), Alternating Directions Implicit method (ADI) (Andersen & Andersen [12]), Front-fixing method & Penalty method (Nielsen & et al. [96]), Backward difference formula BDF2 (Oosterlee [100]), Operator splitting method (Ikonen & Toivanen [71]), Explicit-implicit finite difference method (Cont & Voltchkova [31]), hybrid finite difference method (Cen & et al. [29]), high-order front-tracking finite difference method (Toivanen [122]), exponential time integration (ETI) method (Tangman & et al. [119]), Higher Order Compact (HOC) (Kumar & et al. [79]), are discretization methods in solving the PDE models aroused in financial market.

1.3.2 NON- DISCRETIZATION TECHNIQUES

These techniques include, Adomian Decomposition Method (ADM) proposed by Adomian [5], Variational Iteration Method (VIM) proposed by He [59], Homotopy Perturbation Method (HPM) proposed by Liao [83], and Homotopy Analysis Method (HAM) proposed by Liao [87], are non-discretization methods used in solving the Black-Scholes equation. The methods were developed for non-linear and ordinary differential equations (ODEs) and PDEs, it is proved by later researches that it can be applied to partial integro-differential equations subject to the satisfaction of initial boundary conditions.

The above discretization and non-discretization methods were developed for solving the nonlinear PDEs. However, further researchers have tested them for linear PDEs with successful results in different fields of engineering and science.

1.4 PRESENT STUDY

The present study "Stochastic Behavior of Spot and Future Commodity Prices: Numerical Methods Approach" focusses primarily on the PDE models of linear (commodities market) and non-linear (securities market) and is based on the scope for further study, as suggested Esekon [41] “Future work will also involve solving the nonlinear Black-Scholes equation using the hyperbolic tangent (Tanh) method.” The ‘Tanh’ method suggested provides an opportunity to arrive at analytical solutions.

Tan-Coth method, the extension of Tanh method, First Integral Method (FIM), and Sine-Cosine Method are three powerful methods identified to solve non-linear PDEs in the literature. Taking these researches as cue the present study aims to test applicability of FIM, Tan-Coth method, and Sine-Cosine methods to solve non-linear Black-Scholes equation.

At the same time, the present study proposes to consider testing of non-discretization techniques such as ADM, VIM, HPM and HAM. These non-discretization techniques are successfully applied by Allahviranloo [8], on linear Black-Scholes equation which becomes the base of the present study to test its applicability to solve one-factor, two-factor and three-factor commodity price models given by Schwartz [111].

CHAPTER-2

LITERATURE SURVEY

Black, F., & Scholes, M. (1973) [24], developed a parabolic PDE of second order for European price call option under the following assumptions:

i. The stock price monitors generalized wiener process with constant expected rate of return and constant volatility of the stock price.
ii. The short trade of securities with full use of proceeds is allowed
iii. There are no operation costs or taxes. All securities are naturally divisible
iv. There are no dividends through the life of the derivatives
v. There are no riskless arbitrage predictions
vi. Security exchange is continuous
vii. The risk free rate of interest is constant and similar for all maturities

The model obtained is:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf$$

with boundary conditions $f(S,t) = \begin{cases} S-K, & S \geq K \\ 0 & S < K \end{cases}$, $at\ t = T$

and $f(0,t) = 0,\ \ 0 \leq t \leq T$

Solution technique: using the transformation $f(S,t) = \frac{1}{c} e^{-ax-b^2 m}\, \overline{W}(s_K, m)$,

where $s_K = \ln\left(\frac{S}{K}\right); m = \frac{\sigma^2}{2}(T-t); a = \frac{r}{\sigma^2} - \frac{1}{2}; b = \frac{r}{\sigma^2} + \frac{1}{2}$

The Black-Scholes equation has been transformed in to $\frac{\partial \bar{W}}{\partial m} = \frac{\partial \bar{W}^2}{\partial {s_K}^2}$ with the boundary condition $W\left(s_{K_T}, 0\right) = \left(e^{b s_{K_T}} - e^{a s_{K_T}}\right)$

where $s_{K_T} = \ln\left(\frac{S_T}{K}\right)$ when $t = T$ and $S_t = S_T$ and $m = 0$.

This is well recognized one dimensional Heat equation. Black-Scholes solved this equation analytically and following Conclusion(s) were drawn.

- The option value increases continuously as T, r or σ^2 rises. In each case, it approaches a supreme value equal to the stock price.
- Option is more impulsive than the stock.

Brennan, M. J., & Schwartz, E. S. (1976) [26], assumed that on the valuation of options embedded in unit-linked (equity-linked) life insurance products the contract advantage was linked directly to the market value of a reference portfolio-the unit-and the embedded guarantee was almost always a maturity guarantee with some specified absolute amount guaranteed to be paid at maturity. Defined in this way, unit-linked life insurance agreements was priced by some adapted version of Black-Scholes option pricing formula.
The model obtained is

$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf$ with the following conditions:

i. At termination T = t

$$f\left(S(t), 0, g_b(t)\right) = \max[S(t) - g_b(t), 0] \quad at\ t = T$$

ii. At any time at which a influence is deemed to be made to the reference portfolio

$$f\left(S(T^-), t - T^-, g_b(t)\right) = f(S(T^+), t - T^+, g_b(t))$$

iii. At any time previous to maturity

$$\lim_{S(t)\to\infty} f_S(S(T), t - T, g_b(t)) = 1$$

iv. When there are no further contributions to be made to the mentioned portfolio $f\left(0, t-T, g_b(t)\right) = 0$

v. At any time previous to the final contribution to the reference portfolio, $f_T\left(0, t-T, g_b(t)\right) = r\, f(0, t-T, g_b(t))$

Solution technique: solved using the finite difference scheme

Conclusion(s):

- The put premium rises with the age of the purchaser at admission essentially less is likely to be the operative term of the policy, and of course this effect is more pronounced for longer-term policies which take the policyholder into the years of high mortality.
- Increased the supposed variance rate from 0.01864 to 0.04 and presented that an increase in the variance rate increases the value of a call option and must therefore decrease the value of a put option.
- Measured the ratio between the amount actually invested in the reference portfolio under the riskless strategy, and the amount deemed to be invested in the reference portfolio, and shown that this ratio at different stages in the contract life assuming different rates of return on funds deemed to be financed in the reference portfolio.

Brennan, M. J., & Schwartz, E. S. (1977) [27], developed a second order parabolic type PDE for American put options in the similar lines of Black-Scholes model and used this model to evaluate the pricing of put contracts traded in the New York dealer market. Solution of the PDE represents the value of the put option and the put option values found using Finite difference techniques.

Model: $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf$ with the following conditions

i. $f(S, T) = \max[K_T - S, 0]$

ii. $f(S, t) \geq \max[K_t - S, 0]$

iii. $f(S,t) \leq K_t$

iv. $f(S,t) \geq 0$

v. $\lim_{S\to\infty} f_S(S,t) = 0$

vi. $f(S,T^-) = \max[(K_{T^-}) - S, f(S - D(T),\ T^+)]$

Solution technique: solved the above problem using finite difference scheme

Conclusion(s):

- The model scientifically over-value the put contracts comparative to the observed market prices

Brennan, M. J., & Schwartz, E. S. (1978) [28], suggested a log transformation of the Black-Scholes PDE to obtain the PDE with constant coefficients which makes it possible to apply the explicit finite difference methods such as Crank-Nicolson method.

Model: $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf$

Solution technique: using the transformation $s = \ln(S)$; $W(s,t) = f(S,t)$ the model reduced to $\frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial s^2} + (r - \frac{\sigma^2}{2})\frac{\partial W}{\partial s} + \frac{\partial W}{\partial t} = rW$, PDE with constant coefficients which has been solved by implicit-explicit finite difference (Crank-Nicolson method) schemes.

Conclusion(s):

- The implicit finite difference estimate to the log transform of the Black-Scholes PDE is also comparable to approximating the diffusion process by jump process. Jump process is indiscriminate one which allows for the possibility that the stock price will jump to infinity of possible future values.
- The simpler explicit finite difference approximation agrees to a three point jump process while the more complex implicit finite difference

approximation corresponds to a generalized jump process to infinity of possible points.

Schwartz, E. S. (1997) [111], offered three models of commodity prices which he named as one-factor, two-factor and three-factor models respectively, and derived the corresponding formulas for pricing futures contract in each model. In the first model he adopts that the logarithm of the spot price of the commodity follows a mean reverting process of the Ornstein-Uhlenbeck type, for the second model, he incorporated a second stochastic factor, the convenience yield in the first model which is mean-reverting and absolutely correlated with the spot price and was further extended the second model by seeing the stochastic interest rates to derive the third model.

One-factor model equation:

$$\frac{{\sigma_1}^2}{2}x^2u_{xx} + k\big(\mu - \tilde{\lambda} - \ln(x)\big)xu_x - u_t = 0$$

with terminal boundary condition $u(x,0) = x$

The two-factor model equation:

$$\frac{\sigma_1^2}{2}x^2u_{xx} + \sigma_1\sigma_2\rho_1 xu_{xy} + \frac{\sigma_2^2}{2}u_{yy} + (r-y)xu_x + \big[k(\alpha - y) - \tilde{\lambda}\big]u_y - u_t = 0$$

with terminal boundary condition $u(x,y,0) = x$

The three-factor model equation:

$$\frac{\sigma_1^2}{2}x^2u_{xx} + \frac{\sigma_2^2}{2}u_{yy} + \frac{\sigma_3^2}{2}u_{zz} + \sigma_1\sigma_2\rho_1 xu_{xy} + \sigma_2\sigma_3\rho_2 u_{yz} + \sigma_1\sigma_3\rho_3 xu_{xz} +$$

$$(z-y)xu_x + k(\hat{\alpha} - y)u_y + a(m^* - z)u_z - u_t = 0$$

with terminal boundary condition $u(x,y,z,0) = x$

where $\hat{\alpha} = \alpha - \frac{\tilde{\lambda}}{k}$

The normal Black-Scholes postulation of stock diffusion model with constant volatility was keenly observed by market participants until 1987's market bang. The crash introduced a new era of market discipline and witnessed to use different volatilities which lead a model through Partial Integro-Differential equation for European option prices which was developed by **Andersen, L., & Andersen, J. (2000) [12]**

Model: $\frac{\partial f}{\partial t} + \left(r - q_d - \lambda_j . \widetilde{m}\right) S \, \frac{\partial f}{\partial S} + \frac{1}{2} J^2(t,S).S^2 \frac{\partial^2 f}{\partial S^2} + \lambda_j \, E[\Delta f] = rf$

$E[\Delta f(t,S)] = E[\Delta f(t,J(t)S)] - f(t,S) = \int_0^{\infty} f(t,Sz) \, \tilde{\xi}(z;t) dz - f(t,S)$

with the condition: $f(S_T, T) = \max(S_T - K, 0)$

where $\{J(t)\}_{t \geq 0}$ is the sequence of positive stochastic variables, and $\widetilde{m} = E[J(t) - 1]$

Solution technique: solved the PIDE using ADI (Alternating Directions Implicit) method.

Conclusion(s):

- They have given the framework for adding Poisson jumps to the standard DVF (Deterministic volatility Function) diffusion models of stock price evolution
- Applied the above PIDE model to the S&P500 market results in a largely constant diffusion volatility overlaid with a substantial jump component

Jensen, B., JØrgensen, P. L., & Grosen, A. (2001) [75], extended the Brennan, M. J., & Schwartz, E. S. [26] model by considering the unit-linked (equity-linked) life insurance products contain a surrender option and with the involvement of any excess return (surplus) generated by the investments—i.e. a bonus option, and obtained the second order PDE

Model: $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial P} = rf$ with the condition $f_T = P(T)$ for European style and $f_S \geq P(S)$ for American style for $0 \leq S < T$

Solution technique: solved using the Finite difference explicit method techniques.
Conclusion(s):

- The participating policies can be extremely sensitive to changes in the time to maturity, variations in the spread between the guaranteed interest rate and the market interest rate, and to modifications in the investment policy (volatility).

Nielsen, B. F., Skavhaug, O., & Tveito, A. (2002) [96], introduced two numerical methods for solving Black-Scholes model of American options in the free and moving boundary.
Model:

$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf \quad for\ S > \bar{S}(t)$ and $0 \leq t < T$ with the following conditions

$f(S,t) = \max(K-S,0)$ at $t = T$ for $S \geq 0$

$\left.\frac{\partial f}{\partial S}\right|_{S=\bar{S}(t)} = -1$

$f(S,t)|_{S=\bar{S}(t)} = K - \bar{S}(t)$

$\lim_{S\to\infty} f(S,t) = 0$

$f(S,t) = K - S$ for $0 \leq S < \bar{S}(t)$

$f(S,t) \geq \max(K-S,0) \quad \forall\, S \geq 0,$ and $0 \leq t < T$ where $\bar{S}(t)$ is called free boundary

Solution technique:

- Solved using front-fixing method. The simple idea of this method is to remove the moving boundary by change of variables, in turns out that this methodology leads to a nonlinear problem defined on a fixed domain. This nonlinear problem has been explained by implicit and upwind explicit difference schemes.
- Penalty method- the basic idea of this technique to add a penalty term to the above problem there by obtained a nonlinear PDE defined on fixed domain. This nonlinear problem has been explained by implicit and upwind explicit difference scheme.

Conclusion(s):

- Computational effectiveness of the schemes differ considerably
- Due to limitations on time steps of upwind explicit scheme, the explicit scheme is much slower than implicit methods.

Oosterlee, C.W. (2003) [100], was replaced the supposed constant volatility with stochastic volatility and achieved a generalization of the Black-Scholes PDE as two dimensional PDE.

Model:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left[S^2\vartheta\frac{\partial^2 f}{\partial S^2} + 2\bar{\rho}\gamma\vartheta S\frac{\partial^2 f}{\partial S\partial\vartheta} + \gamma^2\vartheta\frac{\partial^2 f}{\partial\vartheta^2}\right] + rs\frac{\partial f}{\partial S} + \left[\alpha_1(\alpha_2 - \vartheta) - \tilde{\lambda}\gamma\sqrt{\vartheta}\right]\frac{\partial f}{\partial\vartheta} -$$

$rf = 0$ with boundary conditions

$$f(0,\vartheta,t) \geq \max(K,0), \qquad \forall\, \vartheta \geq 0, t \in [0,T]$$

$$f(S,0,t) \geq \max(K - S,0), \quad \forall S \geq 0, t \in [0,T]$$

Solution Technique: solution was achieved with the help of backward difference formula BDF2

Conclusion(s):

- They choose Crank-Nicolson scheme (also called trapezoid rule) discretization, because of its L-stability eccentric and having more advantageous damping properties.
- The time discretization exactness of this implicit scheme is second order.
- With the acceleration technique, fast convergence is attained for an option pricing problem on grids with different grid sizes. The error of the discretization is determined by evaluation with reference solutions.

Ikonen, S., & Toivanen, J. (2004) [71], transformed the generalized Black-Scholes PDE of Oosterlee, C. W. [100], to a linear complementarity problem with initial and boundary conditions.

Solution technique: solved this problem using operator splitting method, in this each time step is divided into two fractional time steps. In the first step a system of linear equations were solved while in the second step the early exercise constraint was prescribed by performing a simple update.

Conclusion(s):

- Studied the accuracy of the operators splitting methods in the numerical experiments and found out that their exactness was similar to the exactness of the PSOR method.
- The splitting does not essentially raise the error.
- The computed prices were in good arrangement with the prices available in the literature.
- The time convergence of the Crank-Nicolson method was somewhat irregular while the time convergence for the L-stable BDF2 and Runge-Kutta methods was sturdier.

Cont, R., & Voltchkova, E. (2005) [31], extended the jump-diffusion model with finite jump intensity given by Andersen, L., & Andersen, J. [12], by considering infinite jump intensity (i.e., singular integral kernels) and developed the following model.

In addition, they suggested an analysis of the convergence of the model which was lacking in Andersen, L., & Andersen, J. [12].

Model:

$$\frac{\partial f}{\partial \tau} = L^{*}f \;\; on\ (0,T]\text{x}\ \mathbb{R}, \quad u(0,s_0) = h(s_0), x \in \mathbb{R}; \quad u(\tau, s_0) = \tilde{g}(\tau, s_0), \quad s_0 \notin \mathbb{R}$$

where

$$L^{*}f = \frac{\sigma^2}{2}\left[\frac{\partial^2 f}{\partial {s_0}^2} - \frac{\partial f}{\partial s_0}\right] + \int_{-\infty}^{+\infty} \nu \, \left[f(s_0 + \varsigma) - f(s_0) - (e^{\varsigma} - 1)\frac{\partial f}{\partial s_0}\right] \, d\varsigma$$

$s_0 = \ln\left(\frac{S}{S_0}\right)$ and $\tau = T - t$, S_0 initial stock price

Solution technique: solved using explicit-implicit finite difference scheme

Conclusion(s):

- When the number of time/space steps is amplified. The performance is quite similar to the case of Black-Scholes model
- The performance of the error (for a fixed grid size) as a function of maturity for a smooth one (forward contract) and a non-smooth one (put option).
- A non-smooth initial condition leads to a lack of small T
- Numerical convergence of a double barrier put price as the number N of space steps rises.

Rodrigoa, M. R., & Mamon, R. S. (2006) [108], developed a model for the price of an option on a time dependent dividend-paying equity.

Model:

$$\frac{\sigma^2}{2} S^2 \, \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial s} + [r - D(t)]S\frac{\partial f}{\partial S} - rf = 0$$

with the conditions $f(S,T) = (S_T - K)^{+} \equiv \max(S_T - K, 0)$

Solution technique:

Using the transformations $f(S,t) = h_0(t)\bar{f}(\bar{S},\bar{t}),\ \bar{S} = \phi_0(t)S,\ \bar{t} = \psi_0(t)$, $\bar{t} = \bar{T}$ when $t = T$, the above model has transformed to PDE with constant coefficients

$$\frac{1}{2}\sigma_c^2\bar{S}^2\frac{\partial^2 f}{\partial\bar{S}^2} + \frac{\partial\bar{f}}{\partial\bar{t}} + r_c\bar{S}\frac{\partial\bar{f}}{\partial\bar{S}} = r_c\bar{f}$$

with terminal condition $\bar{f}(\bar{S},\bar{T}) = \max(\bar{S}_T - K, 0)$

It was solved analytical in the similar lines of Black-Scholes model

Conclusion(s):

- Results indicates that the price of a European call option on a non-dividend paying equity is decomposed as a product of three simple terms involving of a Black–Scholes price for the constant-coefficient case in a non-dividend-paying set-up, the ratio of two strike prices, and a modified factor reflecting the parametrised time.
- This offered method can also be applied to other European-type options such as puts

Cen, Z., Le, A., & Xi, L. (2007) [29], were applied hybrid finite difference scheme on a piecewise uniform mesh for a class of Black-Scholes equations governing option pricing which is path-dependent.

Model:

$\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} + [rS - D(t)]\frac{\partial f}{\partial S} = rf \qquad for\ (S,t) \in \Omega$ with the following conditions

$$f(0,t) = g_1(t);\ \ f(S_T,t) = g_2(t),\ t\epsilon[0,T);\quad f(S,T) = g_3(S),\ S \in \mathbb{R}_0$$

Solution technique:

- Solved using hybrid finite difference scheme on a piecewise uniform mesh. In spatial discretization a hybrid finite difference scheme linking a central difference method with an upwind difference method on a piecewise uniform mesh was used.

- For time discretization, they used an implicit difference method on a uniform mesh.

Conclusion(s):

- On applying the discrete maximum principle and barrier function technique they proved that their scheme was second-order convergent in space for the arbitrary volatility and the arbitrary asset price.
- For K = 1024 a sufficiently large special value they obtained second-order convergence in space.

Xi, L., Cen, Z., & Chen, J. (2008) [132], presented a numerical method combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization to solve Black-Scholes PDE.

Model: they considered the model obtained by Cen, Z., Le, A., & Xi, L. [29]

Solution technique: solved by combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization.

Conclusion(s):

- The difference scheme is steady for the arbitrary volatility and arbitrary asset price.
- They showed that the scheme was second-order convergent with respect to both time and spatial variables
- This difference scheme can handle the degeneracy of the Black-Scholes differential operator at S = 0 without truncating the domain

Company, R., Jodar, L., & Pintos, J. R. (2009) [30], developed non-linear Black-Scholes equation model for European vanilla call option pricing under transaction costs.

Model:

$\frac{1}{2}\sigma_0^2\left(1+\varphi_0\left[\exp\left(r(T-t)a_0^2S^2\frac{\partial^2 f}{\partial S^2}\right)\right]\right)S^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}+\frac{\partial f}{\partial t}=rf\ ,\ S>0,$

$t\in[0,T]$

with following conditions

$f(S,t)|_{t=T}=\max(0,S-K)\,,S>0;\quad f(0,t)=0;\quad \lim_{S\to\infty}\frac{f(S,t)}{S-K\,e^{-r\,(T-t)}}=1$

where $a_0=\mu_0\sqrt{\gamma_0 N_0}$

$\sigma_n^2=\sigma_0^2\left(1+\varphi_0\left[\exp\left(r(T-t)a_0^2S^2\frac{\partial^2 f}{\partial S^2}\right)\right]\right).$

Solution technique:

The above nonlinear problem transformed into another simpler nonlinear parabolic problem with bounded domain.

Transformation:

Using the substitution $s_n=e^{r\,(T-t)}S;\ \tau_n=\frac{\sigma_0^2}{2}(T-t);\ U=e^{r\,(T-t)}f$, the above problem reduced to $\frac{\partial U}{\partial\tau_n}-\left[1+\varphi_0\left(a_0^2s_n^2\frac{\partial^2 U}{\partial s_n^2}\right)\right]s_n^2\frac{\partial^2 U}{\partial s_n^2}=0;\ 0<s_n<\infty,\ 0<\tau_n\leq\frac{\sigma_0^2\,T}{2}$ with initial-boundary conditions

$U(0,\tau_n)=0;\quad \lim_{s_n\to+\infty}U(s_n,\tau_n)=s_n-K;\quad U(s_n,0)=\max(0,s_n-K)$

The transformed partial differential equation has been solved using an explicit finite difference scheme.

Conclusion(s):

- The solution of scheme is positive, monotonically increasing in the space-index
- The parameter a_0 has a direct influence in the steadiness condition
- For $a_0=0$, the model becomes well known Black-Scholes PDE

- The numerical scheme was reliable i.e.: the exact hypothetical solution of the partial differential equation approximates well to the exact solution of scheme as the step sizes tends to zero

Jump-diffusion mathematical models lead to partial integro-differential operators that are non-local, owed to the integral part. That their discretization yields full matrices makes various methods computationally too expensive. Andersen, L., & Andersen, J. [12], and Ikonen, S., & Toivanen, J. [71], have considered numerical methods for jump-diffusion mathematical models based on the linear complementarity problem and variational inequality formulations, through finite difference discretization. One of the central objectives of those studies has to advance computational efficiency by using second-order exact discretizations and faster ways to handle the integral operator. **Toivanen, J. (2010) [122],** derived a numerical method based on the free-boundary formulation for pricing American options in jump-diffusion models with finite jump activity. For easiness, he considered only American put options; similar methods can easily be resultant as well for American call options when the underlying asset paying dividends constantly. His front-tracking method achieves an implicit finite difference discretization on time-dependent non-uniform grids refined near the expiry and free boundary. For interpolations amongst grids and the construction of finite difference stencils, Lagrange interpolation polynomials were used. It gives an easy way to implement fourth-order accurate discretization as well. A non-linear system of equations is solved using Brent's root-finding method, which is easy to use, robust and efficient at each time step. An improvement of that formulation is that it was easy to develop higher-order methods by tracking the location of the free boundary and then by refining grids sufficiently near the free boundary, where the solution is less steady. Also, suggested second-order and fourth-order perfect discretization with respect to the number of time and space steps. The numerical tests confirmed that these convergence rates are attainable under the Black–Scholes model.

Model:

$$\frac{\partial f}{\partial t} = -\frac{1}{2}(\sigma S)^2 \frac{\partial^2 f}{\partial S^2} - \left(r - \tilde{\lambda}\xi_0\right)S\,\frac{\partial f}{\partial S} + \left(r + \tilde{\lambda}\right)f - \tilde{\lambda}\int_{\mathbb{R}^+} f\left(t, Ss_j\right) f_0\left(s_j\right)ds_j$$

$(t, S) \in [0, T)\text{x}\mathbb{R}^+$

If $\tilde{\lambda} = 0$, the above model will reduces to the standard Black-Scholes PDE [24]

Solution technique:

Solved based on the free-boundary formulation for pricing American options under jump-diffusion models with finite jump activity. For simplicity, he considered only American put options; similar methods can easily be derived as well for American call options when the underlying asset paying dividends continuously. His front-tracking method performs an implicit finite difference discretization on time-dependent non-uniform grids refined near the expiry and free boundary.

For interpolations between grids and the construction of finite difference stencils, Lagrange interpolation polynomials were used.

Conclusion(s):

- An advantage of this formulation is that it is easy to develop higher-order methods by tracking the location of the free boundary and then by refining grids sufficiently.
- This gives an easy way to implement fourth-order accurate discretization as well.
- At each time step, a non-linear system of equations is solved using Brent's root-finding method, which is easy to use, robust and efficient near the free boundary, where the solution is less regular.

Sophocleous, C., & Leach, P. G. L. (2010) [115], solved the commodity price models analytically

Model: considers the one-factor, two-factor, and three-factor commodity price models given by Schwartz, E. S. [111]

Solution technique: solved these equations using Lie point symmetries and obtained analytical solution

Tangman, D. Y., & et al. (2011) [119], developed a model for pricing fixed-strike arithmetic Asian options under the Black–Scholes model and obtained the following model

Model:

$$\frac{\partial \bar{f}}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \bar{f}}{\partial S^2} + (r - q_d)S\frac{\partial \bar{f}}{\partial S} - r\bar{f} + \left(\frac{S-a_s}{T-t}\right)\frac{\partial \bar{f}}{\partial a_s} \quad 0 \le S < \infty, \;\; 0 \le a_s < \infty, \;\; 0 \le t \le T$$

with the following conditions

$$\bar{f}(S, a_s, 0) = \max[\emptyset_0(a_s - K), 0]\,; \; \frac{\partial \bar{f}}{\partial t} = -r\bar{f} \text{ as } S \to 0; \;\; \frac{\partial^2 \bar{f}}{\partial S^2} = 0 \text{ as } S \to \infty$$

where $\emptyset_0 = 1$ for fixed strike call and $\emptyset_0 = -1$ for fixed strike put

Solution technique:

- The methodology uses the exponential time integration (ETI) scheme in combination with a dimensional splitting technique. They have chosen to implement this time stepping scheme though ETI can be expensive over very refined meshes.

Conclusion(s):

- Showed that precise Asian option prices can be obtained by using dimensional splitting, which involves a spectral or a central discretization in the asset price, a Hermite interpolation beside the average quantity and a Strang splitting strategy within the ETI framework developed.
- They have designated how to obtain at least second-order convergent solutions for Asian options with multiple features using the Black–Scholes model, the jump-diffusion model and CGMY processes.

Yun, T. (2011) [136], developed a model for price changing of commodities under the assumption that the rising price of commodity immediately effect the price of its relying products without any delay.

Model:

$$\frac{\partial u}{\partial t} = p_c \frac{\partial^2 u}{\partial x^2}$$

where; $t < t_0$ is an equilibrium state

The initial-boundary conditions are:

$(i) u_j(0, 0^-) = u_{j0} \;\; at \;\; t = t_0 = 0 \; and \; u_j = 0$

$(ii) \frac{\partial u_j}{\partial t} = u_j(0, 0^+) - u(0, 0^-) = \dot{u}_{j0} \;\; at \;\; t = t_0 = 0 \; and \; u_j = 0$

$(iii) \frac{\partial u}{\partial x} = \frac{[u(x,0^-) - u(0,0^-)]}{x-0} = u'_{j0} \;\; at \;\; t = t_0 = 0 \; and \; u_j = 0$

Solution technique: solved the above problem using the substitution $u(x,t) = A\, e^{c_1 x + c_2 t}$ and hence obtained the solution is $u(x,t) = u_{j0} e^{\left(\frac{u'_{j0}}{u_{j0}}\right) x_j + \left(\frac{\dot{u}_{j0}}{u_{j0}}\right) t}$

Conclusion(s):

- Equivalence to heat diffusion equation, the price changing diffusion equation was obtained via the description of Newton's second law
- The major dissimilarity between the above equation and heat diffusion equation was that the constant can be measured and is known as a given constant in heat equation, while herein the constant p_c is tough to be measured and is treated to be an unknown constant and has given by $p_c = \frac{u_{j0} \dot{u}_{j0}}{\left(u'_{j0}\right)^2}$
- When the price varying declines then the substitution can be replaced by $u(x,t) = A\, e^{c_1 x - c_2 t}$ to get the solution.

Fadugba, S., Nwozo, C., & Babalola, T. (2012) [48], they underwent to the comparative study of the convergence of the two numerical methods to the Black-Scholes price of European options.

Model:

Considered the standard Black-Scholes PDE [24]

Conclusion(s):

- Both the numerical methods have its advantages and disadvantages of use: finite difference method converges faster and more accurate, they are fairly robust and good for pricing vanilla option. They can also require sophisticated algorithms for solving large sparse linear systems of equations and are relatively difficult to code.
- Monte Carlo method works effectively for pricing both European and exotic options, it is flexible in handling varying and even high dimensional financial problems, hence in spite of its significant progress, an early exercise is problematic.
- Crank Nicolson method is unconditionally steady, more précised and converges faster than Monte Carlo method when pricing European option.

Nwozo, C. R., & Fadugba, S. E. (2012) [98], they underwent to the comparative study of the convergence of the three methods to the Black-Scholes price of European options.

Model: Considered the standard Black-Scholes PDE [24] and Binomial model

Solution technique: Monte Carlo method and Finite difference method (Crank-Nicolson) and with the results of Binomial model

Conclusion(s):

- Binomial models are good for pricing options with early exercise opportunities and they are relatively easy to implement but can be quite tough to adjust to more complex functions.

- Finite difference methods converge quicker and more accurate; they are fairly tough and good for pricing vanilla options where there are possibilities of early exercise.
- Monte Carlo method works perfectly for pricing European options, approximates every arbitrary exotic options, it is flexible in handling varying and even high dimensional financial problems.
- Crank Nicolson method is unconditionally steady, more perfect and converges faster than Binomial model and Monte Carlo method when pricing European option.
- Monte Carlo method is good for pricing the path dependent options

Kumar, A., Waiko, A., & Chakrabarty, S. P. (2012) [79], developed a model for the pricing of arithmetic average strike Asian call option and obtained the following model:

Model:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + S\frac{\partial f}{\partial I} = rf$$

where $f(S, I, t)$=Asian call option price; $I(t) = \int_0^t S(\varsigma)\, d\varsigma$

The above problem is three dimensional which leads to greater computational expenses. This inspires the reduction of higher dimension problem into lower dimension.

Transformation:

$$\bar{R}(t) = \frac{1}{S(t)}\int_0^t S(u)du$$

Let $f(S, I, t) = S\,\bar{H}(\bar{R}, t)$ for some function $\bar{H}(\bar{R}, t)$ by substituting this the above equation reduces to

$\frac{1}{2}\sigma^2 \bar{R}^2 \frac{\partial^2 \bar{H}}{\partial \bar{R}^2} + (1 - r\bar{R})\frac{\partial \bar{H}}{\partial \bar{R}} + \frac{\partial \bar{H}}{\partial t} = 0$, with the following conditions:

$$\bar{H}(\bar{R}(T), T) = \max\left(1 - \frac{1}{T}\bar{R}(T), 0\right)$$

$\bar{H}(\bar{R},t) = 0 \ \ for \ \bar{R} \rightarrow \infty$

$\frac{\partial \bar{H}}{\partial t} + \frac{\partial \bar{H}}{\partial \bar{R}} = 0$ as $\bar{R} \rightarrow 0$

Solution technique:

Crank-Nicolson Implicit Method (CNIM) and Higher Order Compact (HOC) which is fourth order finite difference scheme

Conclusion(s):

The results attained by both the methods were excellent agreement with Monte Carlo results

For very small values of σ, the results attained using HOC Scheme are more accurate compared with the Crank-Nicolson Implicit method.

Yun, T. (2012) [137], studied the application of the instant diffusion equation to the calculation of strategy on changing of owning shares or currencies. The strategy of selling share(s) with maximum altering rate of price-ratio and purchasing share(s) with lowest altering rate of price-ratio (SMaPMi) was calculated by instant diffusion equation with multiple sources of stock-price changing.

Model: considered the same model of Yun, T. [136]

where $u_j(x,t)$= represents the price of commodity 'x' at time 't' due to a raising price changing source at x_j; The diffusion with beginning at time 0^- and end at the time 0^+ changes an old equilibrium state to a new equilibrium state.

(i) $u_j(x_j, 0^-) = u_{j0} \ \ at \ \ t = 0^-$

(ii) $\frac{\partial u_j}{\partial t} = \frac{[u_j(x_j,0^+) - u_j(x_j,0^-)]}{\Delta t} = \frac{[u_j(x_j,0^+) - u_j(x_j,0^-)]}{1} = \dot{u}_{j0} \ \ at \ t = 0^-$

(iii) $\frac{\partial u_j}{\partial x} = \lim_{x \rightarrow x_j} \frac{[u_j(x,0^-) - u_j(x_j,0^-)]}{x - x_j} = u'_{j0} \ \ at \ t = 0^-$

Solution technique: solved the above problem using the substitution $u_j(x,t) = A \ e^{c_{j1}(x-x_j)+c_{j2}t}$ and hence obtained the solution is

$$u_j(x,t) = u_{j0} e^{\left(\frac{u'_{j0}}{u_{j0}}\right)(x-x_j)+\left(\frac{\dot{u}_{j0}}{u_{j0}}\right)t}$$

Conclusion(s):

- SMaPMi is well-matched for short term speculation, if operator is proper.
- Diffusion is a process from the beginning of a breaking of an old stability state to the end of a new stability state due to inertia. The calculation of approach of SMaPMi based on diffusion equation of multiple sources, was suited for time t ≥ 0+ (the end of the new equilibrium state) if no new breaking facto acting.
- SMaPMi is also suited for changing of currencies.

Esekon, J. E. (2013) [41], studied the hedging of derivatives in illiquid markets and derived a nonlinear Black-Scholes equation given by

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\left(1 + 2\rho_l\, S\frac{\partial^2 f}{\partial S^2}\right) + \frac{1}{2}\rho_l(1-\alpha_b^2)\sigma^2 S^4 \left(\frac{\partial^2 f}{\partial S^2}\right)^3 + rS\frac{\partial f}{\partial S} - rf = 0$$

If $\alpha_b = 1, r > 0$ this corresponds to no slippage and the model moderates to

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\left(1 + 2\rho_l\, S\frac{\partial^2 f}{\partial S^2}\right) + rS\frac{\partial f}{\partial S} - rf = 0$$

If $r = 0$ then the model reduces to

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\left(1 + 2\rho_l\, S\frac{\partial^2 f}{\partial S^2}\right) = 0$$

If $\rho_l = 0$ then the asset's price monitors the standard Black-Scholes model with constant volatility

Solution technique: assumed the solution of a forward wave, a classical solution for the nonlinear Black-Scholes equation was found.

Conclusions:

The solution of this model supports all the suppositions of Black-Scholes model [24], that the option is more volatile than the stock.

Allahviranloo, T., & Behzadi, Sh. (2013) [8], solved the standard Black-Scholes, using non-discretization techniques.

Model: considered the standard Black-Scholes [24] equation

Solution Technique: solved the Black-Scholes equation using Adomian decomposition method, Modified Adomian decomposition method, Variational iteration method, Modified Variational iteration method, Homotopy perturbation method, Modified Homotopy perturbation method, and Homotopy analysis method.

Conclusion(s): Homotopy analysis method was the faster convergent method than the other considered methods.

2.1 CONCLUSION

In the literature it is found that Black-Scholes (BS) equation (used to find option price in securities market) has been solved by discretization techniques such as Finite Difference and Finite Element Methods and non-discretization techniques such as Adomian Decomposition Method, Variational Iteration Method, Homotopy Perturbation Method and Homotopy Analysis Method.

It has also been observed that solution of the commodity price models (used to find future prices for commodity products) have been solved using discretization techniques. In these techniques the calculations become cumbersome and these techniques are quite difficult to handle by market traders. It has been observed that modeling of spot and future commodity price models have been used in the international commodity products. Very few Indian commodity products are covered under these models.

2.2 OBJECTIVE

In this investigation we shall try to solve the nonlinear Black Scholes equation (Esekon [41]) using the analytical methods like (i) First Integral Method, (ii) Tanh-Coth Method and (iii) Sine-Cosine Method. These methods are found to be powerful in solving nonlinear partial differential equations (NPDE).

We shall also study the applicability of approximate solution techniques like (i) Adomian Decomposition Method, (ii) Variational Iteration Method, (iii) Homotopy Perturbation Method and (iv) Homotopy Analysis Method to solve PDE equations occurring in commodity price models. These techniques give results what are in close agreement with the exact solutions. These techniques would be easy to understand and apply at market traders' level.

CHAPTER-3

SOLUTION TECHNIQUES

These methods are going to be used in the following chapters.

3.1 ADOMIAN DECOMPOSITION METHOD (ADM)

Adomian decomposition method was introduced by Adomian [5], and in the literature it is found that it is effective and reliable for applying ordinary (Biazar & et al. [20], Fadugba [46], Fadugba & et al. [47]), partial differential equations of linear (Ali [7], Bohner [25], Lesnic [81]), non-linear (Adomian [6], Behirya & et al. [16], El-Wakil & et al. [40], Ghoreishi & et al. [52], Luo & et al. [89], Montazeri [93]), and fractional differential equations (Safari & Danesh [110], Wu & et al. [131]). In the literature it is also found that modified ADM (Eltayeb [39], Hasan & Zhu [57], Hosseini [69], Jiaoa & et al. [76], Vahidi & Kordshouli [123].

The Adomian decomposition method is applied to the following general non-linear equation

$$Lw + Rw + Nw = \tilde{\tilde{f}} \tag{3.1}$$

where u is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is the remaining linear differential operator of order less than L, Nw represents the nonlinear terms and $\tilde{\tilde{f}}$ is the source term.

Applying the inverse operator L^{-1} to both sides of equation (3.1) we obtain

$$w(.) = \varphi + L^{-1}[\tilde{\tilde{f}}] - L^{-1}[Rw] - L^{-1}[Nw] \tag{3.2}$$

where φ the constant of integration is satisfies the condition $L\varphi = 0$

Assume that the solution w can be represented as infinite series of the form

$$w = \sum_{n=0}^{\infty} w_n \quad (3.3)$$

Furthermore, suppose that the non-linear term $Nw = G(w)$ can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$Nw = \sum_{n=0}^{\infty} A_n \quad (3.4)$$

where A_n are Adomian polynomials (Biazar & Shafiof [19]) of Nw can be determined formally as follows

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left\{N\left(\sum_{i=0}^{\infty}\lambda^i w_i\right)\right\}\right]_{\lambda=0} \quad (3.5)$$

The first three Adomian polynomials are:

$$A_0 = G(w_0)$$

$$A_1 = \left[\frac{d}{d\lambda}G(w_0 + w_1\lambda)\right]_{\lambda=0} = w_1 G'(w_0)$$

$$A_2 = \frac{1}{2}\left[\frac{d}{d\lambda}\{(w_1 + 2w_2\lambda)G'(w_0 + w_1\lambda)\}\right]_{\lambda=0} = w_2 G'(w_0) + \frac{w_1^2}{2!}G''(w_0)$$

$$A_3 = \frac{1}{3}\left[\frac{d}{d\lambda}\left\{(w_2 + 3w_3\lambda)G'(w_0 + w_1\lambda) + \frac{(w_1+2w_2\lambda)^2}{2!}G''(w_0 + w_1\lambda)\right\}\right]_{\lambda=0}$$

$$= w_3 G'(w_0) + w_1 w_2 G''(w_0) + \frac{w_1^3}{3!}G'''(w_0) \quad (3.6)$$

Substituting (3.3)-(3.6) in (3.2) gives

$$\sum_{n=0}^{\infty} w_n = \varphi + L^{-1}\left[\widehat{f}\right] - L^{-1}\left[R\left\{\sum_{n=0}^{\infty} w_n\right\}\right] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right] \quad (3.7)$$

3.2 VARIATIONAL ITERATION METHOD (VIM)

The Variational Iteration Method (VIM), proposed by He ([59], [60]) and was applied successfully to autonomous ordinary differential equations (Khader [77]), nonlinear systems of partial differential equations (Abdou [4], Bildik & Konuuralp [23], Duangpithak [34], Duangpithak & Torvattanabun [35], Ganji & et al. [50], Ghorbani & Saberi-Nadjafi [51], Golbaba & Javidi [53], He [61], [62], He [63], [65], He & Wu [66], [67], Javidi & Golbabai [74], Li [82], Liu & et al.

[88], Momani & Abuasad [92], Sadighi & Ganji [109], Soliman [113], Soliman & Abdou [114], Tatari & Dehghan [121], Wazwaz [128]-[130], Yusufoglu [138]), nonlinear differential equations of fractional order (Abbasbandy [1], Inc [72], Odibat [99]) and integro-differential equations (Sweilam [116], Xu [134], Yousefi & et al. [135]).

Consider the following differential equation

$$Lw + Nw = \tilde{\tilde{f}} \tag{3.8}$$

where L is linear operator, N is nonlinear operator and $\tilde{\tilde{g}}$ is a known real function. According to VIM, we can construct a correction functional, $w(.)$ as follows:

$$w_{n+1} = w_n(.) + \int_0^t \phi \left\{ Lw_n(.) + N\widetilde{w_n}(.) - \tilde{\tilde{f}}(.) \right\} dt \tag{3.9}$$

where ϕ is the general Lagrange multiplier, w_0 is an initial approximation, $\widetilde{w_n}$ is the restricted variation, i.e. $\delta\widetilde{w_n} = 0$. The optimal value of the general Lagrange multipliers ϕ can be identified by using the stationary conditions of the variational theory.

For sufficiently large values of n we can consider w_n as an approximation of the exact solution.

3.3 HOMOTOPY PERTURBATION METHOD (HPM)

Homotopy Perturbation Method was proposed by He [64] and the detailed applications of this method can be found in Liao [83].

Let us consider the nonlinear differential equation,

$$\mathcal{A}(w) - \tilde{\tilde{f}}(\boldsymbol{r}) = 0, \qquad \boldsymbol{r} \in \Omega \quad \text{with boundary conditions,} \tag{3.10}$$

$$B\left(w, \frac{\partial w}{\partial \eta}\right) = 0, \qquad \boldsymbol{r} \in \Gamma \tag{3.11}$$

where $\mathcal{A}$ is differential operator, B is boundary operator, $\tilde{\tilde{f}}(\boldsymbol{r})$ is an analytic function, Γ is the boundary of the domain Ω.

The operator, $\mathcal{A}$ has been divided in to two parts, L and N, where L is linear, and N is nonlinear.

Equation (3.10) can be written as, $L(w) + N(w) - \overset{\frown}{f}(\boldsymbol{r}) = 0$ (3.12)

Construct a homotopy $v_h(r,p): \Omega \times [0,1] \rightarrow \mathbb{R}$ which satisfies

$$\mathcal{H}(v_h, p) = (1-p)[L(v_h) - L(w_0)] + p\left[\mathcal{A}(v_h) - \overset{\frown}{f}(\boldsymbol{r})\right] = 0, \quad (3.13a)$$

or

$$\mathcal{H}(v_h, p) = L(v_h) - L(w_0) + p\,L(w_0) + p\left[N(v_h) - \overset{\frown}{f}(\boldsymbol{r})\right] = 0, \quad (3.13b)$$

where $\boldsymbol{r} \in \Omega$, $p \epsilon [0,1]$ is an embedding parameter, w_0 is an initial approximation of equation (3.10), which satisfies the boundary conditions. From equation (3.13), we obtain

$$\mathcal{H}(v_h, 0) = L(v_h) - L(w_0) = 0,$$

$$\mathcal{H}(v_h, 1) = \mathcal{A}(v_h) - \overset{\frown}{f}(\boldsymbol{r}) = 0,$$

The process of changing p from zero to unity is just that of $v_h(\boldsymbol{r}, p)$ from $w_0(\boldsymbol{r})$ to $w(\boldsymbol{r})$.

This process is called deformation in topology, $L(v_h) - L(w_0)$, and $\mathcal{A}(v_h) - \overset{\frown}{f}(\boldsymbol{r})$ are called homotopic.

From equation (3.13), it can be written as,

$$v_h = v_{h_0} + p v_{h_1} + p^2 v_{h_2} + \cdots \quad (3.14)$$

For $p = 1$, the approximate solution of equation (3.10),

$$w = \lim_{p \to 1} v_h = v_{h_0} + p v_{h_1} + p^2 v_{h_2} + \cdots \quad (3.15)$$

The present homotopy perturbation method is obtained with the combination of perturbation method and the homotopy method. The rate of convergence of homotopy perturbation method depends upon the nonlinear operator $\mathcal{A}(v_h)$.

3.4 HOMOTOPY ANALYSIS METHOD (HAM)

Homotopy Analysis Method was proposed by Liao [84], and given a detailed explanation about the choice of convergence control parameter by Liao [87]. The applicability of HAM was found in the literature (Abbasbandy [2], Abbasbandy & Jalili [3], Alomari & et al. [9], Das & et al. [32], Dinarvand & et al. [33], Esmail & Habibolla [43], Ezzati & Aqhamohamadi [44], Fadravi & et al. [45], Gupta & Gupta [56], Hashmi et al. [58], Liao [86], Mustafa [94], Nik & Shateyi [97]).

Consider the differential equation $N[w(.)] = 0$ (3.16)

where N is called nonlinear operator, w is an unknown function in the independent variables

Construct a zeroth-order deformation equation,

$$(1-q)L[\emptyset_h(.\ ;q) - w_0] = q c_0\, H(.)\, N[\emptyset_h(.\ ;q)] \tag{3.17}$$

where $q \in [0,1]$ is the embedding parameter, $c_0 \neq 0$ is called convergence control parameter, L is called an auxillary linear operator, $\emptyset_h(.\ ;q)$ is called an unknown function, w_0 is an initial guess of $w(.)$ and $H(.)$ represents a non-zero auxillary function. For $q = 0$ and $q = 1$ from the equation (3.17) we obtain,

$\emptyset_h(.\ ;0) = w_0, \quad \emptyset_h(.\ ;1) = w(.)$ respectively.

As q increases from 0 to 1, the solution $\emptyset_h(.\ ;q)$ varies from w_0 to $w(.)$. On expanding the function $\emptyset_h(.\ ;q)$ in Taylor's series with respect to q we obtain,

$$\emptyset_h(.\ ;q) = w_0(.) + \sum_{m=1}^{\infty} w_m(.) q^m \tag{3.18}$$

where $w_m(.) = \frac{1}{m!}\left(\frac{\partial^m \emptyset_h(.\ ;q)}{\partial q^m}\right)_{q=0}$ (3.19)

The convergence of (3.18) depends upon the convergence control parameter c_0. If convergence is obtained at $q = 1$ we obtain,

$$w(.) = w_0 + \sum_{m=1}^{\infty} w_m(.) \tag{3.20}$$

Differentiate the equation (3.17), m-times with respect to q and the dividing them by $m!$ and by setting $q = 0$, we obtain the mth-order deformation equation:

$$L[w_m(.) - \chi_m w_{m-1}(.)] = c_0\, R_m(\vec{w}_{m-1}) \tag{3.21}$$

where $R_m(\vec{w}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\emptyset_h(.\ ;q)]}{\partial q^{m-1}}\Big]_{q=0}$ and $\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$ (3.22)

3.5 FIRST INTEGRAL METHOD (FIM)

First Integral Method was proposed by Feng [49] and many of the researchers shown its efficacy in various fields tested by Biazar & Aslanpanah [18], El-Ganaini [37], El-Sabbagh & El-Ganaini [38], Eslami & et al. [42], Hosseini & et al. [68], Raslan [107], Sharma & Kushel [112], Taghizadeh [117], Taghizadeh & et al. [118], Tascan & Bekir [120].

Consider the nonlinear partial differentia equation $\bar{F}\left(w, w_t, w_\varsigma, w_{\varsigma\varsigma}, w_{\varsigma t}, \ldots\right) = 0,$

(3.23)

where $w(.)$ is the solution of the above equation (3.23). Let us use the transformations

$w(\varsigma, t) = \widetilde{\widehat{w}}\,(\xi),\ \ \xi = \varsigma - ct$ where $'c'$ is constant (3.24)

Using chain rule, we obtain

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.),\ \frac{\partial}{\partial \varsigma}(.) = \frac{d}{d\xi}(.),\quad \frac{\partial^2}{\partial \varsigma^2}(.) = \frac{d^2}{d\xi^2}(.), \ldots \tag{3.25}$$

On Substituting (3.25) in (3.23), we obtain the ODE

$$\widetilde{\widehat{G}}\left(\widetilde{\widehat{w}}, \widetilde{\widehat{w}}_\xi, \widetilde{\widehat{w}}_{\xi\xi}, \ldots\right) = 0 \tag{3.26}$$

On, introducing new independent variables $X(\xi) = \widetilde{\widehat{w}}\,(\xi), Y = \widetilde{\widehat{w}}_\xi\,(\xi)$ (3.27)

Using (3.26) and (3.27), we obtain a system of ODEs

$$\begin{cases} X_\xi(\xi) = Y(\xi) \\ Y_\xi(\xi) = F_1\left(X_\xi(\xi), Y_\xi(\xi)\right) \end{cases} \tag{3.28}$$

Using the qualitative theory of ordinary differential equations, then the general solutions to (3.28) can be obtained directly. Though, in general, it is very difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. So, we apply the Division Theorem to obtain one first integral to (3.28) which reduces (3.26) to a first order integrable ODE. Then, an exact solution to (3.23) is obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(.)$ and $Q(.)$ are polynomials of two variables in $\mathbb{C}[.]$ and $P(.)$ is irreducible. If $Q(.)$ vanishes at all zero points of $P(.)$, then there exists a polynomial $G_d(.)$ in $\mathbb{C}[.]$ such that

$Q(.) = P(.)\ G_d(.)$

3.6 Tanh-Coth METHOD

The extension of Tanh method is called Tanh-Coth method proposed by Bekir & Cevikel [17]. In the literature it is found that both tanh and its extension methods are powerful in solving the nonlinear partial differential equations (Alquran [11], Baldwin & et al. [14], Baldwin [15], El-Borai & et al. [36], Gozukizil & Akcagil [54], Lee & Sakthivel [80], Malfliet [90], Parkes & Duffy [104], Wazwaz [127], Zayed & Abdelaziz [140]).

Let us consider a nonlinear equation $\bar{F}\left(w, w_t, w_\varsigma, w_{\varsigma\varsigma}, w_{\varsigma t}, \dots\right) = 0,$ (3.29)

Let us assume that $\xi = \varsigma - ct$ where $'c'$ is constant, so that $w(.) = \widehat{w}\,(\xi)$

which implies, $\frac{\partial}{\partial t} = -c\frac{d}{d\xi}, \frac{\partial}{\partial \varsigma} = \frac{d}{d\xi},\ \frac{\partial^2}{\partial \varsigma^2} = \frac{d^2}{d\xi^2},\ \frac{\partial^3}{\partial \varsigma^3} = \frac{d^3}{d\xi^3}, \dots$ (3.30)

On substituting (3.30) in the equation (3.29), we obtain

$\widehat{G}\left(\widehat{w}\,, \widehat{w}_\xi\,, \widehat{w}_{\xi\xi}\,, \dots\right) = 0$ which is an ODE (3.31)

If every term of equation (3.31) has derivatives with respect to ξ, then by integrating this equation, and by assuming that the constant of integration is zero, we obtain a simplified ODE.

Let us consider $Y = \tanh(\xi)$ (3.32)

Using (3.32) the derivatives in right hand side of (3.30) will leads to,

$\frac{d}{d\xi} = (1 - Y^2)\frac{d}{dY}$

$$\frac{d^2}{d\xi^2} = -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2}$$

$$\frac{d^3}{d\xi^3} = 2(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6Y(1-Y^2)^2\frac{d^2}{dY^2} + (1-Y^2)^3\frac{d^3}{dY^3} \qquad (3.33)$$

Let $U(\xi)$ is in the form,

$$\tilde{\tilde{w}}(\xi) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} \frac{b_k}{Y^k} \qquad (3.34)$$

where M is an positive integer, will be determined by equating the powers of Y. In most of the cases, M will be positive integer, otherwise a transformation formula will be used to overcome this difficulty.

3.7 Sine-Cosine METHOD

Sine-Cosine method is the extension of the Sine-function method and it was proposed by Alquran & Al-Khaled [11]. It is proved in the literature that it is effectively applicable in various fileds of engineering and science to get the exact solution of ordinary and partial differential equations (Alquran [10], Arbabi & Abadi [13], Bibi & Mohyud-Din [21], [22], Jaafar & Jawad [73], Guner & et al. [55], Najafi & et al. [95], Rab & Akhter [105], Wazwaz [126], Xie & Tang [133], Zayed & Abdelazi [139]).

Let us consider a nonlinear equation $\bar{F}\left(w, w_t, w_\varsigma, w_{\varsigma\varsigma}, w_{\varsigma t}, \ldots\right) = 0,$ (3.35)

Let us assume that $\xi = \varsigma - ct$ where $'c'$ is constant, so that $w(.) = \tilde{\tilde{w}}(\xi)$

which implies, $\frac{\partial}{\partial t} = -c\frac{d}{d\xi}, \frac{\partial}{\partial \varsigma} = \frac{d}{d\xi}, \frac{\partial^2}{\partial \varsigma^2} = \frac{d^2}{d\xi^2}, \frac{\partial^3}{\partial \varsigma^3} = \frac{d^3}{d\xi^3}, \ldots$ (3.36)

On substituting (3.36) in the equation (3.35), we obtain

$\tilde{\tilde{G}}\left(\tilde{\tilde{w}}, \tilde{\tilde{w}}_\xi, \tilde{\tilde{w}}_{\xi\xi}, \ldots\right) = 0$ which is an ODE (3.37)

Integrate the equation (3.37) as many times as possible with respect to ξ, and setting the constant of integration is zero, we obtain a simplified ODE.

Let us consider the solution is of the form

$$w(.) = \begin{cases} \lambda \sin^{\beta}(\mu_s \xi) \\ \quad or \\ \lambda \cos^{\beta}(\mu_s \xi) \end{cases} \tag{3.38}$$

where λ, β, μ_s are constants need to be determined.

Using (3.38) the derivatives in right hand side of (3.36) will leads to,

$w(\xi) = \lambda \sin^{\beta}(\mu \xi)$,

$w^n(\xi) = \lambda^n \sin^{n\beta}(\mu_s \xi)$,

$\left(w^n(\xi)\right)_{\xi} = n\mu_s \beta \lambda^n \cos(\mu_s \xi) \sin^{n\beta-1}(\mu_s \xi)$,

$$\left(w^n(\xi)\right)_{\xi\xi} = -n^2\mu_s^2\beta^2\lambda^n \sin^{n\beta}(\mu_s \xi) + n\mu_s^2\lambda^n\, \beta(n\beta - 1) \sin^{n\beta-2}(\mu_s \xi) \tag{3.39a}$$

or

$w(\xi) = \lambda \cos^{\beta}(\mu_s \xi)$,

$w^n(\xi) = \lambda^n \cos^{n\beta}(\mu_s \xi)$,

$\left(w^n(\xi)\right)_{\xi} = -n\mu_s \beta \lambda^n \sin(\mu_s \xi) \cos^{n\beta-1}(\mu_s \xi)$,

$$\left(w^n(\xi)\right)_{\xi\xi} = -n^2\mu_s^2\beta^2\lambda^n \cos^{n\beta}(\mu_s \xi) + n\mu_s^2\lambda^n\, \beta(n\beta - 1) \cos^{n\beta-2}(\mu_s \xi) \tag{3.39b}$$

where $\lambda, \mu, \beta,$ and c, parameters need to be determined. Substitute the equation (3.39a) or (3.39b) in (3.37), and balancing the exponents of the trigonometric functions cosine or sine, collecting all the terms with same power in $\cos^{n\beta}(\mu_s \xi)$ or $\sin^{n\beta}(\mu_s \xi)$ and equate their coefficients to zero we obtain a system of algebraic equations among the unknowns λ, μ_s, β and c. These values will be determined using computerized symbolic calculations.

3.8 TECHNICAL COMPUTING SOFTWARE

To apply ADM, VIM, and HAM techniques we developed specially designed codes in **MATLAB** programming language**.**

We also have used the well-developed codes in **Maple** to verify the solutions using Tanh-Coth method and Sine-Cosine method.

CHAPTER-4

BLACK-SCHOLES EQUATION AND ITS SOLUTION

Let us consider the nonlinear Black-Scholes equation (Esekon [41]) with

$\alpha_b = 1, r > 0$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\left(1 + 2\rho_l\, S \frac{\partial^2 f}{\partial S^2}\right) + rS\frac{\partial f}{\partial S} - rf = 0 \qquad (4.1)$$

Differentiate the equation (4.1) twice with respect to S, we obtain

$$\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2}(1 + 4\rho_l Sg)\frac{\partial^2 g}{\partial S^2} + 2\rho_l \sigma^2 S^3 \left(\frac{\partial g}{\partial S}\right)^2 + 2\sigma^2 S(1 + 6\rho_l Sg)\frac{\partial g}{\partial S} + rS\frac{\partial g}{\partial S}$$

$$+\sigma^2(1 + 6\rho_l Sg)g + rg = 0 \qquad (4.2)$$

where $g = \frac{\partial^2 f}{\partial s^2}$

Using the transformation $g = \frac{\bar{g}}{\rho_l S}$ and $s = \ln(S)$, the equation (4.2) reduces to

$$\frac{\partial \bar{g}}{\partial t} + \frac{\sigma^2}{2}(1 + 4\bar{g})\frac{\partial^2 \bar{g}}{\partial s^2} + 2\sigma^2\left(\frac{\partial \bar{g}}{\partial s}\right)^2 + \frac{\sigma^2}{2}(1 + 4\bar{g})\frac{\partial \bar{g}}{\partial s} + r\frac{\partial \bar{g}}{\partial s} = 0 \qquad (4.3)$$

Again using the transformation $\bar{g} = \frac{V-1}{4}$, the equation (4.3) further reduces to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\left\{\frac{\partial}{\partial s}\left(V\ \frac{\partial V}{\partial s} + \frac{1}{2}V^2 + \frac{2r}{\sigma^2}\ V\right)\right\} = 0 \qquad (4.4)$$

Equation (4.4) is a nonlinear PDE with constant coefficients.

4.1 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING FIRST INTEGRAL METHOD

Let us consider the transformation $\xi = s - ct$ where $'c'$ is constant

Using (3.25)-(3.28), the equation (4.4) can be written as,

$$\frac{\sigma^2}{2} X Y_\xi + \frac{\sigma^2}{2} Y^2 + (r-c)Y + \frac{\sigma^2}{2} XY = 0 \tag{4.5}$$

where $X = V(\xi)$ and $Y = V_\xi$

$$X_\xi = Y \tag{4.6a}$$

$$Y_\xi = \frac{2}{\sigma^2}\frac{1}{X}\left[-\frac{\sigma^2}{2} XY - (r-c)Y - \frac{\sigma^2}{2} Y^2\right] \tag{4.6b}$$

Let us suppose that $d\xi = X\, d\varpi$, then the equations (4.6a) and (4.6b) becomes

$$X_\varpi = XY \tag{4.7a}$$

$$Y_\varpi = -XY - k_\sigma (r-c)Y - Y^2 \tag{4.7b}$$

where $k_\sigma = \frac{2}{\sigma^2}$

Let us assume that $X = X(\varpi)$ and $Y = Y(\varpi)$ are non-trivial solutions of equations (4.7a), (4,7b) and

$P(X, Y) = \sum_{i=0}^{m} a_i(X) Y^i$ is an irreducible polynomial in the complex domain

$\mathbb{C}[X, Y]$ such that

$$P\big(X(\varpi), Y(\varpi)\big) \equiv \sum_{i=0}^{m} a_i\big(X(\varpi)\big)\, Y(\varpi)^i = 0 \tag{4.8}$$

where $a_i(X), (i = 0, 1, \dots, m)$ are the polynomials in X and $a_m(X) \neq 0$. Equation (4.8) is called the first integral to equation (4.7a)-(4.7b), due to Division theorem, there exists a polynomial $h_1(X) + h_2(X)Y$ in the complex domain $\mathbb{C}[X, Y]$ such that

$$\frac{dP}{d\varpi} \equiv \frac{\partial P}{\partial X}\frac{\partial X}{\partial \varpi} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \varpi} = (h_1(X) + h_2(X)Y)\left\{\sum_{i=0}^{m} a_i\big(X(\varpi)\big)\, Y(\varpi)^i\right\} \tag{4.9}$$

Suppose that $m = 1$ in the equation (4.8) and compare the coefficients of Y^i $(i = 2,1,0)$ on both sides of (4.9), we obtain

$$X\,a_1' = (h_2 + 1)a_1 \qquad (4.10a)$$

$$X\,a_0' = X\,a_1 + (r - c)\,k_\sigma\,a_1 + h_1 a_1 + h_2 a_0 \qquad (4.10b)$$

$$h_1 a_0 = 0 \qquad (4.10c)$$

From (4.10a), we conclude that $a_1(X)$ is constant and $h_2(X) = -1$. For simplicity, let us consider $a_1(X) = 1$.

From (4.10b), it concludes that $\deg\big(h_1(X)\big) \leq \deg\big(a_0(X)\big)$.

Using $a_1(X)$ and $h_2(X)$ values, from (4.10b), it can be written as,

$$Xa_0' = X + (r - c)k_\sigma + h_1 - a_0 \qquad (4.10d)$$

From (4.10d), it can be concluded that $a_0(X)$ is not a polynomial.

Hence, due to lack of polynomial $a_0(X)$, FIM will not be applied to solve the nonlinear Black-Scholes equation.

4.2 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING Tanh-Coth METHOD

Let us consider the transformation $\xi = s - ct$ where $'c'$ is constant

Using (3.30)-(3.33), the equation (4.4) can be written as

$$\frac{\sigma^2}{2}\,V\frac{\partial^2 V}{\partial \xi^2} + \frac{\sigma^2}{2}\left(\frac{\partial V}{\partial \xi}\right)^2 + (r - c)\frac{\partial V}{\partial \xi} + \frac{\sigma^2}{2}V\,\frac{\partial V}{\partial \xi} = 0 \qquad (4.11)$$

Integrating the equation (4.11) on both sides,

$$\frac{\sigma^2}{2}V\,\frac{\partial V}{\partial \xi} + (r - c)V + \frac{\sigma^2}{4}V^2 = 0 \qquad (4.12)$$

From (4.12), by balancing the nonlinear term $\left(V\,\frac{\partial V}{\partial \xi}\right)$ with the highest order liner term it can be concluded that M will not be a positive integer.

Hence, due to lack of a positive integer value of M, the tanh-coth method will not be applied to solve the nonlinear Black-Scholes equation.

4.3 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING Sine-Cosine METHOD

Let us consider $V = \lambda \cos^{\beta}(\mu_s \xi)$ and from the equation (4.11), we obtain

$$\lambda\beta\mu_s c \cos^{\beta-1}(\mu_s\xi)\ \sin(\mu_s\xi) + \lambda^2\beta^2\mu_s^2\sigma^2\cos^{2\beta-2}(\mu_s\xi) -$$

$$\lambda^2\beta^2\mu_s^2\,\sigma^2\cos^{2\beta}(\mu_s\xi) - \frac{\lambda^2}{2}\beta\mu_s\sigma^2\cos^{2\beta-1}(\mu_s\xi)\sin(\mu_s\xi) -$$

$$\frac{\lambda^2}{2}\beta\mu_s^2\sigma^2\cos^{2\beta-2}(\mu_s\xi) - r\lambda\,\beta\mu_s\cos^{\beta-1}(\mu_s\xi)\sin(\mu_s\xi) = 0 \qquad (4.13)$$

From (4.13), equating the exponents $2\beta - 2$ and $\beta - 1$ yields

$2\beta - 2 = \beta - 1$, so that $\beta = 1$

It needs to be noted that, on equating the exponent pairs $\beta - 1 = 2\beta$ we obtain the same value of $\beta = 1$.

Setting the coefficients to zero yields,

$$\lambda\beta\mu_s c - r\lambda\beta\mu + \lambda^2\beta^2\mu_s^2\sigma^2 - \frac{\lambda^2}{2}\beta\mu_s^2\sigma^2 = 0 \qquad (4.14a)$$

$$\lambda\beta\mu_s c - r\lambda\beta\mu_s - \lambda^2\beta^2\mu_s^2\sigma^2 = 0 \qquad (4.14b)$$

$$-\frac{\lambda^2}{2}\beta\mu_s\sigma^2 = 0 \qquad (4.14c)$$

From (4.14c), we obtain either $\lambda = 0$ or $\mu_s = 0$

In both the cases we obtain the zero solution.

4.4 CONCLUSION

Literature survey shows that nonlinear equations may be solved using Tanh-Coth, Sine-Cosine, and FIM methods. As a part of this study, a particular non-linear Black-Scholes equation (Esekon [41]) is selected to examine its validation using above methods.

The results show that, while balancing of exponents using Tanh-Coth method, the positive integer (M) could not be obtained, hence, Tanh-Coth method is not suitable for its application. Secondly, while using Sine-Cosine method, during balancing exponents, β happens to be 1, that supports application of method, however, the solution happens to be trivial that challenging efficacy of the process of validation. Similarly, during the application of FIM method, nonexistence of integral polynomials happens to be the distinct short coming of the method.

CHAPTER-5

ONE FACTOR COMMODITY PRICE MODEL AND ITS SOLUTION

5.1 ONE FACTOR COMMODITY PRICE MODEL

The One Factor Commodity Price Model equation is

$$\frac{\sigma_1^2}{2}x^2u_{xx} + k\big(\mu - \tilde{\lambda} - \ln(x)\big)xu_x - u_t = 0 \tag{5.1}$$

with the erminal boundary condition $u(x,0) = x$. (5.2)

The closed form solution of the above equation (5.1) along with (5.2) is given by Schwartz [111].

$$u(x,t) = \exp\left[e^{-kt}\, ln(x) + (1 - e^{-kt})\left(\mu - \frac{\sigma_1^2}{2k} - \tilde{\lambda}\right) + \frac{\sigma_1^2}{4k}(1 - e^{-2kt})\right] \tag{5.3}$$

5.2 SOLUTION OF ONE FACTOR COMMODITY PRICE MODEL USING ADM

To obtain the approximate solution to equation (5.1) along with (5.2), according to ADM, we can write as follows

$$u = u(x,0) + L^{-1}\left[\frac{\sigma_1^2}{2}x^2u_{xx} + k\big(\mu - \tilde{\lambda} - \ln(x)\big)xu_x\right] \tag{5.4}$$

$$u_0 = x \tag{5.5}$$

$$u_{n+1} = \int_0^t A_n\, dt \quad \text{for } n \geq 1 \tag{5.6}$$

5.3 SOLUTION OF ONE FACTOR COMMODITY PRICE MODEL USING VIM

Using (3.8)-(3.9), the equation (5.1) can be written as

$u_{n+1} =$

$$u_n(x,t) + \int_0^t \phi \left\{\frac{\partial u_n(x,t)}{\partial t} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n(x,t)}{\partial x^2} - k\big(\mu - \tilde{\lambda} - \ln(x)\big) x \frac{\partial u_n(x,t)}{\partial x}\right\} dt \quad (5.7)$$

$\delta u_{n+1} =$

$$\delta u_n(x,t) + \delta \int_0^t \phi \left\{\frac{\partial u_n(x,t)}{\partial t} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 \widetilde{u_n}(x,t)}{\partial x^2} - k\big(\mu - \tilde{\lambda} - \ln(x)\big) x \frac{\partial \widetilde{u_n}(x,t)}{\partial x}\right\} dt$$

(5.8)

$$\delta u_{n+1} = \delta u_n(x,t) + \delta \int_0^t \phi \left\{\frac{\partial u_n(x,t)}{\partial t}\right\} dt \quad (5.9)$$

$$\delta u_{n+1} = \delta u_n(x,t)(1+\phi) - \delta \int_0^t \phi' \delta u_n(x,t) dt \quad (5.10)$$

which yields the stationary conditions

$$1 + \phi = 0,\ \phi' = 0 \Longrightarrow \phi = -1 \quad (5.11)$$

Substituting the value of $\phi = -1$ into the functional (5.7) give the iteration formulas

$$u_{n+1} = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,t)}{\partial t} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n(x,t)}{\partial x^2} - k\big(\mu - \tilde{\lambda} - \ln(x)\big) x \frac{\partial u_n(x,t)}{\partial x}\right\} dt$$

(5.12)

5.4 CONVERGENCE OF SOLUTION OF ONE FACTOR COMMODITY PRICE MODEL

5.4.1 Convergence of solution of One Factor Commodity Price Model using VIM has been verified as given in the following theorem developed in our investigation. (Pannala [101])

Let us consider the functions defined as $F_{11} \equiv x^2 u_{xx}(x,t)$, $F_{12} \equiv x u_x(x,t)$, and $F_{13} \equiv x \ln x \, u_x(x,t)$ are Lipschitz continuous with $|F_{11}(u) - F_{11}(u^*)| \leq L_{11}|u - u^*|$, $|F_{12}(u) - F_{12}(u^*)| \leq L_{12}|u - u^*|$ and $|F_{13}(u) - F_{13}(u^*)| \leq L_{13}|u - u^*|$

Theorem 5.1: The solution $u_n(x,t)$ obtained from (5.12) converges to the solution of problem (5.1) when $0 < \beta_{11} < 1$ and $0 < \beta_{12} < 1$ where $\beta_{11} = \{|d_{11}|L_{11} + |d_{12}|L_{12} + |k|L_{13}\}$ and $\beta_{12} = \{1 - T(1 - \beta_{11})\}$

Proof: $u_{n+1} = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,t)}{\partial t} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n(x,t)}{\partial x^2} - k\big(\mu - \tilde{\lambda} - \ln(x)\big) x \frac{\partial u_n(x,t)}{\partial x}\right\} dt$

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,t)}{\partial t} - d_{11}F_{11}(u_n) - d_{12}F_{12}(u_n) + kF_{13}(u_n)\right\} dt \quad (5.13)$$

where $d_{11} = \frac{\sigma_1^2}{2}$ and $d_{12} = k\big(\mu - \tilde{\lambda}\big)$

$$u_n(x,t) = u(x,t) - \int_0^t \left\{\frac{\partial u(x,t)}{\partial t} - d_{11}F_{11}(u) - d_{12}F_{12}(u) + kF_{13}(u)\right\} dt \quad (5.14)$$

Let $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - u(x,t)$, $|e_n(x,t^*)| = \max_t |e_n(x,t)|$. Since e_n is a decreasing function with respect to $'t'$ from (5.13), (5.14) and mean value theorem we obtained,

$$e_{n+1}(x,t) = e_n(x,t) + \int_0^t \left[\frac{\partial(-e_n)}{\partial t} - d_{11}\{F_{11}(u_n) - F_{11}(u)\} - b\{F_{12}(u_n) - F_{12}(u)\} + k\,\{F_{13}(u_n) - F_{13}(u)\}\right] dt$$

$$e_{n+1}(x,t) \leq e_n(x,t) + \int_0^t (-e_n) dt + \{|d_{11}|L_{11} + |d_{12}|L_{12} + |k|L_{13}\} \int_0^t |e_n| dt$$

$$e_{n+1}(x,t) \leq e_n(x,t) - Te_n(x,\omega) + \{|d_{11}|L_{11} + |d_{12}|L_{12} + |k|L_{13}\} \int_0^t |e_n| dt$$

$e_{n+1}(x,t) \leq e_n(x,t) - Te_n(x,\omega) + \{|d_{11}|L_{11} + |d_{12}|L_{12} + |k|L_{13}\}T\ |e_n(x,t)|$

$e_{n+1}(x,t) \leq \{1 - T(1-\beta_{11})\}\ |e_n(x,t^*)|$

where $0 \leq \omega \leq t$, hence $e_{n+1}(x,t) \leq \beta_{12}|e_n(x,t^*)|$, therefore,

$\|e_{n+1}\| = \max_{\forall t\in J}|e_{n+1}| \leq \beta_{12} \max_{\forall t\in J}|e_n| \leq \beta_{12}\|e_n\|$

Since, $0 < \beta_{12} < 1$, then $\|e_n\| \to 0$.

5.4.2 Convergence of solution of One Factor Commodity Price Model using ADM

Define $\eta_i = \begin{cases} \frac{\|u_{i+1}\|}{\|u_i\|} & \text{for } \|u_i\| \neq 0 \\ 0 & \text{for } \|u_i\| = 0 \end{cases}$ if $0 \leq \eta_i < 1,\ i = 1,2,3,\ldots$ then according to Hosseini [70], $\sum_{i=0}^{\infty} u_i$ converges to the exact solution u.

5.5 NUMERICAL EXAMPLES

Example 5.1: Consider $\sigma_1 = 1,\ k = 1,\ \mu = 1.2\ and\ \tilde{\lambda} = 1$ in the equation (5.1) Then using (5.4)-(5.6), we obtain the following approximants

$u_0 = x$

$u_1 = -t * x * \left(log(x) - \frac{1}{5}\right)$

$u_2 = \frac{t^2 * x * (30 * log(x) + 50 * log(x)^2 - 33)}{100}$

$u_3 = \frac{t^3 * x * (5 * log(x) + 14) * (20 * log(x) - 50 * log(x)^2 + 23)}{1500}$

$$u_4 = \frac{t^4 * x * (1600 * log(x)^2 - 22280 * log(x) + 13000 * log(x)^3 + 2500 * log(x)^4 - 2591)}{60000} \quad (5.15)$$

Adding all the approximants in (5.15) we obtain the approximate solution of (5.1) for n=4, as

$$u(x,t) = \Big[x - x*\ t*\left(log(x) - \frac{1}{5}\right) + \frac{x*t^2*(50*log(x)^2 + 30*log(x) - 33)}{100} + \frac{x*t^3*(5*log(x) + 14)*(20*log(x) - 50*log(x)^2 + 23)}{1500} + \frac{x*t^4*(2500*log(x)^4 + 13000*log(x)^3 + 1600*log(x)^2 - 22280*log(x) - 2591)}{60000}\Big] \quad (5.16)$$

Example 5.2: Solved the example 5.1 using VIM

We obtain the following approximant with four iterations

$$u_4(x,t) = \left(\frac{t^2*(50*\log(x)^2 + 30*\log(x) - 33)}{100} - t*\left(\log(x) - \frac{1}{5}\right) + \frac{t^4*(2500*\log(x)^4 + 13000*\log(x)^3 + 1600*\log(x)^2 - 22280*\log(x) - 2591)}{60000} + \frac{t^3*(5*\log(x) + 14)*(20*\log(x) - 50*\log(x)^2 + 23)}{1500} + 1\right)*x$$

5.6 SOLUTION OF ONE FACTOR COMMODITY PRICE MODEL USING HAM, AND HPM

5.6.1 SOLUTION USING HAM:

Using $k = .301,\ \mu = 3.093,\ \sigma_1 = .334,\ \tilde{\lambda} = -.242$ we obtain the following polynomial in x, t, c_0 with the help of MATLAB for n=4,

$$u_4 = x*\Big(1.004*c_0*t - 3.012*c_0^2*t + 3.012*c_0^3*t - 1.004*c_0^4*t + 1.033*c_0^2*t^2 - 2.066*c_0^3*t^2 + 0.07858*c_0^3*t^3 + 1.033*c_0^4*t^2 - 0.07858*c_0^4*t^3 - 0.009155*c_0^4*t^4 - 0.5*c_0*t^2*\big(0.01679*c_0 - 1.0*\big(0.301*c_0 + 0.001505*c_0*(200.0*\log(x) - 667.0)\big)*(0.301*\log(x) - 1.004)\big) - 0.301*c_0*t*\log(x) + 0.1359*c_0^2*t^2*\log(x)^2 - 0.2718*c_0^3*t^2*\log(x)^2 + 0.09552*c_0^3*t^3*\log(x)^2 +$$

$$0.1359 * c_0^4 * t^2 * \log(x)^2 - 0.01364 * c_0^3 * t^3 * \log(x)^3 - 0.09552 * c_0^4 * t^3 * \log(x)^2 + 0.01364 * c_0^4 * t^3 * \log(x)^3 + 0.004307 * c_0^4 * t^4 * \log(x)^2 - 0.00251 * c_0^4 * t^4 * \log(x)^3 + 0.000342 * c_0^4 * t^4 * \log(x)^4 + 0.001505 * c_0^2 * t * (200.0 * \log(x) - 667.0) - 4.456e-29 * c_0 * t * (4.506e28 * c_0 + 6.755e27 * \log(x) + 6.755e27 * c_0^2 * \log(x) - 1.546e28 * c_0 * t + 1.546e28 * c_0^2 * t - 1.351e28 * c_0 * \log(x) - 2.253e28 * c_0^2 - 5.879e26 * c_0^2 * t^2 + 1.153e28 * c_0 * t * \log(x) - 7.146e26 * c_0^2 * t^2 * \log(x)^2 + 1.02e26 * c_0^2 * t^2 * \log(x)^3 - 2.033e27 * c_0 * t * \log(x)^2 - 1.153e28 * c_0^2 * t * \log(x) + 2.033e27 * c_0^2 * t * \log(x)^2 + 1.408e27 * c_0^2 * t^2 * \log(x) - 2.253e28) + 0.903 * c_0^2 * t * \log(x) - 0.903 * c_0^3 * t * \log(x) + 0.301 * c_0^4 * t * \log(x) - 0.7706 * c_0^2 * t^2 * \log(x) + 1.541 * c_0^3 * t^2 * \log(x) - 0.1882 * c_0^3 * t^3 * \log(x) - 0.7706 * c_0^4 * t^2 * \log(x) + 0.1882 * c_0^4 * t^3 * \log(x) + 0.003495 * c_0^4 * t^4 * \log(x) - 0.00301 * c_0 * t * (200.0 * \log(x) - 667.0) + 1.0\Big) \quad (5.17)$$

where c_0 is called convergence control parameter.

5.6.2 SOLUTION USING HPM

If $c_0 = -1$, then HAM will be in the form of HPM as proposed by Liao [85].

5.7 RESULTS AND DISCUSSION

5.7.1 RESULTS

The following tables: 5.1 to 5.3, are prepared with various parameter values for Crude Oil data of the model Schwartz [111], using ADM and VIM

Table 5.1 represents the absolute errors obtained against exact Crude oil future prices with $k = .301,\ \mu = 3.093,\ \sigma_1 = .334,\ \tilde{\lambda} = -.242$ for various iterations

x	t	Exact solution	Absolute error for n=4	Absolute error for n=7	Absolute error for n=10	Absolute error for n=15
19.5	0	19.5	3.55E-15	3.55E-15	3.55E-15	3.55E-15
20	0.2	20.39393	2.69E-07	1.71E-12	7.11E-15	7.11E-15
20.5	0.4	21.21941	7.53E-06	7.40E-10	4.05E-13	3.55E-15
21	0.6	21.97683	4.85E-05	2.64E-08	3.54E-11	0
21.5	0.8	22.6676	0.000166741	3.33E-07	8.40E-10	0
22	1	23.29393	0.000390394	2.36E-06	9.62E-09	4.12E-13
22.5	1.2	23.85861	0.000672286	1.16E-05	6.91E-08	7.28E-12
23	1.4	24.36487	0.000803582	4.45E-05	3.57E-07	8.08E-11
23.5	1.6	24.81623	0.000305903	0.000140893	1.44E-06	6.27E-10

Table 5.2 represents the absolute errors obtained against exact Crude oil future prices with $k = .694,\ \mu = 3.037,\ \sigma_1 = .326,\ \tilde{\lambda} = -.072$ for various iterations

x	t	Exact solution	Absolute error for n=4	Absolute error for n=7	Absolute error for n=10	Absolute error for n=15
13	0	13	5.33E-15	5.33E-15	5.33E-15	5.33E-15
13.5	0.2	14.40627	8.60E-06	1.14E-09	1.95E-14	5.33E-15
14	0.4	15.65393	0.000260708	2.14E-07	1.66E-10	3.55E-15

14.5	0.6	16.74017	0.001810017	3.45E-06	2.22E-08	2.03E-12
15	0.8	17.67096	0.006734248	1.47E-05	6.56E-07	2.71E-10
15.5	1	18.45785	0.017465934	2.15E-05	8.53E-06	1.10E-08
16	1.2	19.11531	0.035265405	0.000512081	6.56E-05	2.11E-07
16.5	1.4	19.65897	0.058047464	0.002995401	0.000349	2.37E-06

Table 5.3 represents the absolute errors obtained against exact Crude oil future prices with $k = .428,\ \mu = 2.991,\ \sigma_1 = .257,\ \tilde{\lambda} = .002$ for various iterations

x	t	Exact solution	Absolute error for n=4	Absolute error for n=7	Absolute error for n=10	Absolute error for n=15
16.8	0	16.8	0	0	0	0
17	0.2	17.21419	6.51E-08	2.94E-11	7.11E-15	7.11E-15
17.2	0.4	17.57762	3.92E-06	7.56E-09	7.43E-13	3.55E-15
17.4	0.6	17.89473	4.27E-05	1.93E-07	4.22E-11	1.07E-14
17.6	0.8	18.16992	0.000230914	1.90E-06	4.86E-10	4.26E-14
17.8	1	18.4074	0.00084802	1.11E-05	1.97E-10	2.54E-12
18	1.2	18.61119	0.002437919	4.63E-05	4.37E-08	6.03E-11
18.2	1.4	18.78501	0.005916944	0.000153123	4.60E-07	8.54E-10
18.4	1.6	18.93231	0.012683783	0.000426273	2.92E-06	8.36E-09

Table 5.4 represents the absolute errors obtained against exact Copper future prices with $k = .369,\ \mu = 4.854,\ \sigma_1 = .233,\ \tilde{\lambda} = -.339$ using ADM and VIM for various iterations

x	t	Exact solution	Absolute error for n=4	Absolute error for n=7	Absolute error for n=10	Absolute error for n=15
110	0	110	4.26E-14	4.26E-14	4.26E-14	4.26E-14
115	0.2	118.7028	2.82E-06	3.92E-11	1.42E-14	1.42E-14

120	0.4	126.7789	8.24E-05	5.28E-09	2.29E-12	2.84E-14
125	0.6	134.1946	0.000544	1.39E-08	2.47E-10	5.68E-14
130	0.8	140.9388	0.001879	1.00E-06	6.34E-09	2.84E-14
135	1	147.0183	0.004318	1.21E-05	7.30E-08	3.78E-12
140	1.2	152.4538	0.006994	7.38E-05	4.93E-07	5.80E-11
145	1.4	157.2757	0.006792	0.000312	2.21E-06	4.76E-10
150	1.6	161.5214	0.002931	0.001026	6.82E-06	1.83E-09
155	1.8	165.2324	0.033572	0.002794	1.28E-05	4.12E-09
160	2	168.4521	0.102224	0.006538	1.50E-06	1.12E-07

Table 5.5 Percentage errors obtained using the equation (5.16)

(x, t)		% errors for n=4	% errors for n=10
	Exact solution	in ADM	in ADM
(.1, .1)	0.1266	0.00021254	2.2142e-12
(.2, .3)	0.31435	0.022777	3.4999e-7
(.3, .2)	0.38378	0.00081735	2.0814e-9
(.4, .4)	0.56245	0.030975	8.2698e-6
(.5, .6)	0.71101	0.38093	0.00052207

Table 5.6 Absolute errors obtained using the equation (5.16)

x	t	absolute error for n=3	absolute error for n=10	absolute error for n=14
0.1	0.1	2.69081E-07	2.80331E-15	8.32667E-17
0.2	0.3	7.15988E-05	1.10017E-09	1.10800E-13
0.3	0.2	3.13681E-06	7.98800E-12	1.66533E-16
0.3	0.4	6.62866E-05	1.76082E-08	7.87781E-12
0.4	0.2	4.91557E-06	2.40791E-11	1.66533E-16
0.4	0.5	5.49150E-04	5.25154E-07	3.45767E-10
0.5	0.3	8.85704E-05	2.13180E-09	2.35256E-13
0.5	0.5	1.10803E-03	5.26954E-07	7.82549E-10
0.6	0.4	4.94478E-04	2.55110E-08	2.37945E-11
0.6	0.5	1.45846E-03	2.67268E-07	8.02700E-10

0.7	0.1	6.15411E-07	9.99201E-16	1.11022E-16
0.7	0.4	5.46780E-04	1.13326E-08	1.30052E-11
0.7	0.5	1.59062E-03	1.50586E-07	4.19500E-10
0.8	0.5	1.51617E-03	6.25021E-07	2.37647E-10
0.9	0.5	1.25602E-03	1.07257E-06	1.00793E-09
1	0.5	8.34698E-04	1.43103E-06	1.74121E-09

Table 5.7 represents the time elapsed in seconds for finding the approximate solution of the model with the parameter values used in table 5.1

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=7	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.510444	3.988302	5.639090	7.932666	15.154135
VIM	2.217024	3.320347	5.346506	8.144588	83.682647

Table 5.8 represents the time elapsed in seconds for finding the approximate solution of the model with the parameter values used in table 5.2

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=7	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.926894	4.439972	6.290687	9.814973	17.037331
VIM	2.363203	4.075299	6.020251	9.834339	122.705986

Table 5.9 represents the time elapsed in seconds for finding the approximate solution of the model with the parameter values used in table 5.3

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=7	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25

ADM	2.973792	3.723436	5.571800	8.676009	11.462024
VIM	2.238381	3.630831	5.255295	8.453601	101.359876

Table 5.10 represents the time elapsed in seconds for finding the approximate solution of the model with the parameter values used in table 5.4

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=7	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.636094	3.944166	4.639100	8.956850	13.467820
VIM	2.152425	3.873026	5.489444	8.209186	101.041525

Table 5.11 represents the possible values of convergence control parameter (c_0) obtained from the equation (5.17) with $k = .301,\ \mu = 3.093,\ \sigma_1 = .334,$ $\tilde{\lambda} = -.242$ and various values of x, t for n=4

$x = 20, t = 0.2$	$x = 20.5, t = 0.4$
3.55343130662573 + 0.00000000000000i	3.09821434862397 + 0.00000000000000i
0.965661690096844 + 2.58775600526590i	0.932423922138502 + 2.16472952191210i
0.965661690096844 - 2.58775600526590i	0.932423922138502 - 2.16472952191210i
-1.62252970806677 + 0.00000000000000i	-1.23449602349703 + 0.00000000000000i
$x = 21, t = 0.6$	$x = 21.5, t = 0.8$
2.84607730544686 + 0.00000000000000i	2.66785225818887 + 0.00000000000000i
0.899903640850532 + 1.94284476683101i	0.867949060775257 + 1.79323605456733i

0.899903640850532 - 1.94284476683101i	0.867949060775257 - 1.79323605456733i
-1.04827910361384 + 0.00000000000000i	-0.934995377873612 + 0.00000000000000i
$x = 22, t = 1$	$x = 22.5, t = 1.2$
2.52726947197325 + 0.00000000000000i	2.40932608185694 + 0.00000000000000i
0.836448151734802 + 1.67990640455548i	0.805317967947988 + 1.58806654410742i
0.836448151734802 - 1.67990640455548i	0.805317967947988 - 1.58806654410742i
-0.858599023710498 + 0.00000000000000i	-0.804264052046744 + 0.00000000000000i
$x = 23, t = 1.4$	$x = 23.5, t = 1.6$
2.30648319575950 + 0.00000000000000i	2.21447419023907 + 0.00000000000000i
0.774497446422827 + 1.51033085744685i	0.743941777852800 + 1.44253247301433i
0.774497446422827 - 1.51033085744685i	0.743941777852800 - 1.44253247301433i
-0.764597548817974 + 0.00000000000000i	-0.735463514269374 + 0.00000000000000i

Table 5.12 represents the possible values of c_0 convergence control parameter obtained for $k = .301,\ \mu = 3.093,\ \sigma_1 = .334,\ \tilde{\lambda} = -.242$ using HAM for n=7

$x = 20, t = 0.2$	$x = 20.5, t = 0.4$
2.49255111716484 + 0.803501114442005i	2.29930919166078 + 0.759805435262053i
2.49255111716484 -	2.29930919166078 -

0.803501114442005i	0.759805435262053i
1.34849179913272 + 1.62879049943596i	1.27742444968459 + 1.43238022286521i
1.34849179913272 - 1.62879049943596i	1.27742444968459 - 1.43238022286521i
-0.0930615131831449 + 1.32934673374291i	-0.00765631384857506 + 1.18344964557266i
-0.0930615131831449 - 1.32934673374291i	-0.00765631384857506 - 1.18344964557266i
-0.730923900666381 + 0.00000000000000i	-0.577408845270316 + 0.00000000000000i
$x = 21, t = 0.6$	$x = 21.5, t = 0.8$
2.17863569423147 + 0.743527025865854i	2.08706425544540 + 0.736447478910551i
2.17863569423147 - 0.743527025865854i	2.08706425544540 - 0.736447478910551i
1.22491066634659 + 1.31142376303365i	1.18000299031636 + 1.22004133341375i
1.22491066634659 - 1.31142376303365i	1.18000299031636 - 1.22004133341375i
0.0317681747425665 + 1.09579309937860i	0.0540754250400093 + 1.03016183821819i
0.0317681747425665 - 1.09579309937860i	0.0540754250400093 - 1.03016183821819i
-0.499266626030706 + 0.00000000000000i	-0.449890178965246 + 0.00000000000000i
$x = 22, t = 1$	$x = 22.5, t = 1.2$
2.01113011264470 + 0.733535728504165i	1.94478672124589 + 0.732773143225451i

2.01113011264470 - 0.733535728504165i	1.94478672124589 - 0.732773143225451i
1.13937272208555 + 1.14476788717063i	1.10159463953277 + 1.07986920486001i
1.13937272208555 - 1.14476788717063i	1.10159463953277 - 1.07986920486001i
0.0676779008115005 + 0.975939150920224i	0.0761133585352276 + 0.928640325434158i
0.0676779008115005 - 0.975939150920224i	0.0761133585352276 - 0.928640325434158i
-0.415552424375421 + 0.00000000000000i	-0.390404808628700 + 0.00000000000000i
$x = 23, t = 1.4$	$x = 23.5, t = 1.6$
1.88480460570394 + 0.733139780873444i	1.82929460626767 + 0.734011684458898i
1.88480460570394 - 0.733139780873444i	1.82929460626767 - 0.734011684458898i
1.06594968862019 + 1.02238887489866i	1.03203656955333 + 0.970610672836218i
1.06594968862019 - 1.02238887489866i	1.03203656955333 - 0.970610672836218i
0.0811795515767032 + 0.885991546232223i	0.0839098520854237 + 0.846704165095014i
0.0811795515767032 - 0.885991546232223i	0.0839098520854237 - 0.846704165095014i
-0.371445390971807 + 0.00000000000000i	-0.356947749215963 + 0.00000000000000i

5.7.2 DISCUSSION

This study has undertaken parameters as were available against pre-executed study by Schwartz [111] for crude oil, say, speed adjustment of spot commodity prices (k), drift rate (μ), volatility of spot price of commodity (σ_1), market price of risk ($\tilde{\lambda}$) and were substituted in the one-factor model, and subsequently, crude oil future prices were found.

Prior to the obtained prices, on substitution of parameters, the obtained polynomial (A 1.1.1) and the corresponding coefficients were tested successfully for convergence, and thus approximate solution was derived. On comparison with exact solutions derived from analytical solutions of the existing study, the obtained approximate solution values from both ADM and VIM methods are found to be precisely matching. The obtained errors (tables 5.1 to 5.3) are found to be abridged significantly with increased number of iterations.

On the similar lines, existing study as refereed above, commodity prices for copper were subjected to one-factor model, and this study derived approximate solutions using ADM and VIM methods (A 1.1.2), and as a matter of successful application the obtained values are precisely matching with exact solutions, and the errors (tables 5.4 to 5.7) were significantly abridged with increased number of iterations.

Further, the study has been investigated to negotiable time lapse between the ADM and VIM methods, for both crude oil and copper prices, the observed delay for 10 iterations or n=10, is largely matching with both the methods, but further increase in the number of iterations, the delay particularly in case of VIM found to be approximately 6 times. Detailed delay specific tables were shown in tables 5.8 to 5.10.

The study further evaluated for HAM method on the above refereed study, and thus obtained polynomial in terms of three variables, that are, $x, t,$ and c_0 (5.17).

When the parameters of the refereed study, x and t were substituted in the above polynomial, 8-sets of polynomials were derived (A 1.2.1 to A 1.2.8). However, the study observed that the c_0 value happens to be not consistent among all the obtained 8-sets of polynomials (table 5.11), an impediment while testing validity of HAM.

5.8 CONCLUSION

The validation of ADM, VIM, HAM and HPM methods, were undertaken for deriving approximate solutions of one-factor commodity price model, in the form of polynomials which in turn will be of immense help while efficaciously predicting the future commodity prices at any short interval of time more accurately and with less degree of error. The obtained errors are to be reduced considerably with increased number of iterations. However, computations through VIM are of longer duration processing compared to ADM to obtain the approximate polynomial. Absence of convergence control parameter c_0 is one of the important limitations while solving the one-factor commodity price model using HAM and HPM.

CHAPTER-6

TWO FACTOR COMMODITY PRICE MODEL AND ITS SOLUTION

6.1 TWO FACTOR COMMODITY PRICE MODEL

The Two Factor Commodity Price Model equation is

$$\frac{\sigma_1^2}{2}x^2u_{xx}+\sigma_1\sigma_2\rho_1 xu_{xy}+\frac{\sigma_2^2}{2}\,u_{yy}+(r-y)xu_x+\left[k(\alpha-y)-\tilde{\lambda}\right]u_y-u_t=0 \tag{6.1}$$

with terminal boundary condition $u(x,y,0)=x$ (6.2)

The closed form solution of the above equation (6.1) along with (6.2) is given by Schwartz [111].

$$u(x,y,t)=xexp\left[-y\frac{1-e^{-kt}}{k}+A(t)\right] \tag{6.3}$$

where

$$A(t)=\left(r-\left(\alpha-\frac{\tilde{\lambda}}{k}\right)+\frac{\sigma_2^2}{2k^2}-\frac{\sigma_1\sigma_2\rho_1}{k}\right)t+\frac{\sigma_2^2}{4}\frac{\left(1-e^{-2kt}\right)}{k^3}+\left(\alpha k-\tilde{\lambda}+\sigma_1\sigma_2\rho_1-\frac{\sigma_2^2}{k}\right)\frac{\left(1-e^{-kt}\right)}{k^2} \tag{6.4}$$

6.2 SOLUTION OF TWO FACTOR COMMODITY PRICE MODEL USING ADM

According to ADM, approximate solution of the equation (6.1) along with (6.2) can be written as

$$u = u(x,y,0) + L^{-1}\left[\frac{\sigma_1^2}{2}x^2 u_{xx} + \sigma_1\sigma_2\rho_1 x u_{xy} + \frac{\sigma_2^2}{2}\, u_{yy} + (r-y)xu_x + \right.$$

$$\left.\left\{k(\alpha - y) - \tilde{\lambda}\right\}u_y\right] \tag{6.5}$$

$$u_0 = x$$

$$u_{n+1} = \int_0^t A_n\, dt \ \ \text{for } n \geq 1$$

6.3 SOLUTION OF TWO FACTOR COMMODITY PRICE MODEL USING VIM

To obtain the approximate solution to equation (6.1) along with (6.2), according to VIM, it can be written as follows

$$u_{n+1} = u_n(x,y,t) + \int_0^t \phi \left[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \right.$$

$$\left.(r-y)x\frac{\partial u_n}{\partial x} - \left\{k(\alpha-y) - \tilde{\lambda}\right\}\frac{\partial u_n}{\partial y}\right] dt \tag{6.6}$$

$$\delta u_{n+1} = \delta u_n(x,y,t) + \delta\int_0^t \phi \left[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 \widetilde{u_n}}{\partial x^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial y^2} - \right.$$

$$\left.(r-y)x\frac{\partial \widetilde{u_n}}{\partial x} - \left\{k(\alpha-y) - \tilde{\lambda}\right\}\frac{\partial \widetilde{u_n}}{\partial y}\right] dt \tag{6.7}$$

$$\delta u_{n+1} = \delta u_n(x,y,t) + \delta\int_0^t \phi\left[\frac{\partial u_n}{\partial t}\right] dt \tag{6.8}$$

$$\delta u_{n+1} = \delta u_n(x,y,t)(1+\phi) - \delta\int_0^t \phi'\delta u_n(x,y,t)dt \tag{6.9}$$

This yields the stationary conditions

$$1+\phi = 0,\ \ \phi' = 0 \Longrightarrow \phi = -1 \tag{6.10}$$

Substituting the value of $\phi = -1$ into the functional (6.6) give the iteration formulas

$$u_{n+1} = u_n(x,y,t) - \int_0^t \left[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \right.$$

$$\left.(r-y)x\frac{\partial u_n}{\partial x} - \left\{k(\alpha-y) - \tilde{\lambda}\right\}\frac{\partial u_n}{\partial y}\right] dt \tag{6.11}$$

6.4 CONVERGENCE OF SOLUTION OF TWO FACTOR COMMODITY PRICE MODEL

6.4.1 Convergence of solution of Two Factor Commodity Price Model using VIM has been verified as given in the following theorem developed in our investigation (Pannala & Vipin [102])

Let us consider the functions $F_{21} \equiv x^2 u_{xx}(x,y,t)$, $F_{22} \equiv x u_{xy}(x,y,t), F_{23} \equiv u_{yy}(x,y,t), F_{24} \equiv u_y(x,y,t), F_{25} \equiv x u_x(x,y,t), F_{26} \equiv y\, u_y(x,y,t)$,

$F_{27} \equiv xy\, u_x(x,y,t)$ are Lipschitz continuous with

$|F_{21}(u) - F_{21}(u^*)| \leq L_{21}|u-u^*|, |F_{22}(u) - F_{22}(u^*)| \leq L_{22}|u-u^*|,$

$|F_{23}(u) - F_{23}(u^*)| \leq L_{23}|u-u^*|, |F_{24}(u) - F_{24}(u^*)| \leq L_{24}|u-u^*|,$
$|F_{25}(u) - F_{25}(u^*)| \leq L_{25}|u-u^*|, |F_{26}(u) - F_{26}(u^*)| \leq L_{26}|u-u^*|$
and $|F_{27}(u) - F_{27}(u^*)| \leq L_{27}|u-u^*|$ for $x, y > 0$, and $J = [0,\ T]$ $(T \in \mathbb{R})$.

Theorem 6.1: The solution $u_n(x,y,t)$ obtained from (6.11) converges to the solution of problem (6.1) when $0 < \beta_{21} < 1$ and $0 < \beta_{22} < 1$ where
$\beta_{21} = \{|c_{21}|L_{21} + |c_{22}|L_{22} + |c_{23}|L_{23} + |c_{24}|L_{24} + |r|L_{25} + |k|L_{26} + L_{27}\},$

$\beta_{22} = \{1 - T(1-\beta_{21})\}$, $c_{21} = \frac{\sigma_1^2}{2}$, $c_{22} = \sigma_1\sigma_2\rho_1$, $c_{23} = \frac{\sigma_2^2}{2}$, and $c_{24} = (k\alpha - \tilde{\lambda})$

Proof: From (6.11)

$$u_{n+1} = u_n(x,y,t) - \int_0^t \left[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - (r-y)x\frac{\partial u_n}{\partial x} - \left\{k(\alpha - y) - \tilde{\lambda}\right\}\frac{\partial u_n}{\partial y}\right] dt$$

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial t} - c_{21}\,F_{21}(u_n) - c_{22}F_{22}(u_n) - c_{23}F_{23}(u_n) - c_{24}F_{24}(u_n) - rF_{25}(u_n) + kF_{26}(u_n) + F_{27}(u_n)\right] d\xi \qquad (6.12)$$

$u_n = u - \int_0^t \Big[\frac{\partial u}{\partial t} - c_{21}\, F_{21}(u) - c_{22}F_{22}(u) - c_{23}F_{23}(u) - c_{24}F_{24}(u) -$

$rF_{25}(u) + kF_{26}(u) + F_{27}(u)\Big]\ dt$ (6.13)

Let us consider that, $e_{n+1}(x,y,t) = u_{n+1}(x,y,t) - u_n(x,y,t),$ and

$e_n(x,y,t) = u_n(x,y,t) - u(x,y,t)$

$|e_n(x,y,t^*)| = \max_t |e_n(x,y,t)|$. Since e_n is a decreasing function with respect to $'t'$ from (6.12), (6.13) and mean value theorem we obtained,

$e_{n+1} = e_n + \int_0^t \Big[\frac{\partial(-e_n)}{\partial t} + c_{21}\{F_{21}(u_n) - F_{21}(u)\} + c_{22}\{F_{22}(u_n) - F_{22}(u)\} +$

$c_{23}\{F_{23}(u_n) - F_{23}(u)\} + c_{24}\{F_{24}(u_n) - F_{24}(u)\} + r\{F_{25}(u_n) - F_{25}(u)\} -$

$k\,\{F_{26}(u_n) - F_{26}(u)\} - \{F_{27}(u_n) - F_{27}(u)\}\Big]\, dt$

$e_{n+1} \le e_n + \int_0^t (-e_n)dt + \{|c_{21}|L_{21} + |c_{22}|L_{22} + |c_{23}|L_{23} + |c_{24}|L_{24} + |r|L_{25} +$

$|k|L_{26} + L_{27}\} \int_0^t |e_n| dt$

$e_{n+1}(x,y,t) \le e_n(x,y,t) - Te_n(x,y,\omega) + \{|c_{21}|L_{21} + |c_{22}|L_{22} + |c_{23}|L_{23} +$

$|c_{24}|L_{24} + |r|L_{25} + |k|L_{26} + L_{27}\} \int_0^t |e_n| dt$

$e_{n+1}(x,y,t) \le e_n(x,y,t) - Te_n(x,y,\ \omega) + \{|c_{21}|L_{21} + |c_{22}|L_{22} + |c_{23}|L_{23} +$

$|c_{24}|L_{24} + |r|L_{25} + |k|L_{26} + L_{27}\}\ T\ |e_n(x,y,t)|$

$e_{n+1}(x,y,t) \le \{1 - T(1 - \beta_{21})\}\ |e_n(x,y,t^*)|$

where $0 \le \omega \le t$, hence $e_{n+1}(x,y,t) \le \beta_{22}|e_n(x,y,t^*)|$, therefore,

$\|e_{n+1}\| = \max_{\forall t\in J}|e_{n+1}| \le \beta_{22} \max_{\forall t\in J}|e_n| \le \beta_{22}\|e_n\|$

Since, $0 < \beta_{22} < 1$, then $\|e_n\| \to 0$.

6.4.2 Convergence of solution of Two Factor Commodity Price Model using ADM

It was discussed in section 5.4.2

6.5 NUMERICAL EXAMPLES

Example 6.1: Consider $\sigma_1 = \sigma_2 = \rho_1 = \tilde{\lambda} = \alpha = r = 1\ and\ k = 2$ in the equation (6.1),

We obtain the following approximants using ADM

$u_0 = x$

$u_1 = -t * x * (y - 1)$

$u_2 = \frac{t^2 * x * (y^2 - 1)}{2}$

$u_3 = -\frac{t^3 * x * y * (y^2 + 3 * y - 5)}{6}$

$$u_4 = \frac{t^4 * x * (y^4 + 8 * y^3 - 2 * y^2 - 20 * y + 7)}{24} \tag{6.14a}$$

Adding all the approximants in (6.14a) we obtain the approximate solution of (6.1) for n=4, as

$$u(x,y,t) = x - t * x * (y - 1) + \frac{t^2 * x * (y^2 - 1)}{2} - \frac{t^3 * x * y * (y^2 + 3 * y - 5)}{6} + \frac{t^4 * x * (y^4 + 8 * y^3 - 2 * y^2 - 20 * y + 7)}{24} \tag{6.14b}$$

Example 6.2: Solved example-6.1 using VIM

We obtain the following approximant using VIM for n=4

$$u_4 = \left(\frac{t^2 * (y^2 - 1)}{2!} - t * (y - 1) + \frac{t^4 * (y^4 + 8 * y^3 - 2 * y^2 - 20 * y + 7)}{4!} - \frac{t^3 * y * (y^2 + 3 * y - 5)}{3!} + 1\right) * x$$

6.6 SOLUTION OF TWO FACTOR COMMODITY PRICE MODEL USING HAM, AND HPM

6.6.1 Solution using HAM

Using a set of parameter values of Crude oil $\sigma_1 = 0.393, \sigma_2 = 0.527, \rho_1 = 0.766, r = 0.06, k = 1.876, \alpha = 0.106, \tilde{\lambda} = 0.19$, obtained the following polynomial in $x, y, t, c_0$with the help of MATLAB for n=4

$$u_4 = x * (0.06 * c_0 * t - 0.1348 * c_0^2 * t^2 + 0.08491 * c_0^3 * t^3 - 0.01366 * c_0^4 * t^4 + 0.5 * c_0 * t^2 * (1.156 * c_0 * y - 0.09344 * c_0 + c_0 * (y - 0.06)^2) + 1.554 * c_0^2 * t^2 * y - 0.4294 * c_0^3 * t^3 * y - 0.003265 * c_0^4 * t^4 * y - 2.0 * c_0 * t * (y - 0.06) - 1.0 * c_0 * t * y + 1.5 * c_0^2 * t^2 * y^2 - 1.644 * c_0^3 * t^3 * y^2 - 0.5 * c_0^3 * t^3 * y^3 + 0.3326 * c_0^4 * t^4 * y^2 + 0.279 * c_0^4 * t^4 * y^3 + 0.04167 * c_0^4 * t^4 * y^4 - 2.368e-24 * c_0 * t * (7.037e22 * c_0^2 * t^2 * y^3 + 2.314e23 * c_0^2 * t^2 * y^2 + 6.043e22 * c_0^2 * t^2 * y - 1.195e22 * c_0^2 * t^2 - 4.222e23 * c_0 * t * y^2 - 4.374e23 * c_0 * t * y + 3.793e22 * c_0 * t + 4.222e23 * y - 2.533e22) + 1.0) \quad (6.15)$$

where c_0 is called convergence control parameter.

6.6.2 Solution using HPM

If $c_0 = -1$, then HAM will be in the form of HPM.

6.7 RESULTS AND DISCUSSION

6.7.1 RESULTS

Risk free interest rate is considered as constant $(r) = 0.06$ (Schwartz [111])

The following tables- 6.1 to 6.3 are prepared for the Crude Oil data with various parameter values (Schwartz [111]) using ADM and VIM

Table 6.1 represents the absolute errors obtained against exact Crude oil future prices with $\sigma_1 = 0.393, \sigma_2 = 0.527, \rho_1 = 0.766, r = 0.06, k = 1.876, \alpha = 0.106, \tilde{\lambda} = 0.198$ for various iterations

x	y	t	Exact solution	Absolute error for n=4	Absolute error for n=10	Absolute error for n=15
19.5	0.1	0	19.5	0	0	0
19.5	0.25	0	19.5	0	0	0
19.5	0.4	0	19.5	0	0	0
20	0.1	0.2	19.85612	0.000328	1.14E-09	3.55E-15
20	0.25	0.2	19.36559	0.000164	5.41E-10	5.33E-14
20	0.4	0.2	18.88718	0.000304	3.22E-09	4.26E-14
20.5	0.1	0.4	20.24592	0.009513	2.10E-06	3.02E-10
20.5	0.25	0.4	19.40926	0.004335	1.16E-06	3.34E-09
20.5	0.4	0.4	18.60717	0.009593	6.15E-06	2.42E-09
21	0.1	0.6	20.67204	0.066091	0.000164	2.31E-07
21	0.25	0.6	19.58507	0.027542	0.000103	2.07E-06
21	0.4	0.6	18.55525	0.071336	0.000499	1.41E-06
21.5	0.1	0.8	21.13021	0.256856	0.003563	2.53E-05
21.5	0.25	0.8	19.85732	0.098432	0.002464	0.000195013
21.5	0.4	0.8	18.66112	0.293227	0.011137	0.000126105
22	0.1	1	21.61451	0.728116	0.038288	0.000952528

22	0.25	1	20.19949	0.257768	0.028684	0.006579433
22	0.4	1	18.8771	0.871394	0.12285	0.004057134
22.5	0.1	1.2	22.11942	1.693686	0.264487	0.018316996
22.5	0.25	1.2	20.59228	0.556043	0.211919	0.115977683
22.5	0.4	1.2	19.17057	2.110494	0.86867	0.06853459

Table 6.2 represents the absolute errors obtained against exact Crude oil future prices with $\sigma_1 = 0.374, \sigma_2 = 0.556, \rho_1 = 0.882,\ r = 0.06, k = 1.829, \alpha = 0.184, \tilde{\lambda} = 0.316$ for various iterations

x	y	t	Exact solution	Absolute error for n=4	Absolute error for n=10	Absolute error for n=15
13	0.1	0	13	0	0	0
13	0.25	0	13	0	0	0
13	0.4	0	13	0	0	0
13.5	0.1	0.2	13.39166	0.000244	9.70E-10	5.33E-15
13.5	0.25	0.2	13.05938	0.000174	1.09E-10	3.02E-14
13.5	0.4	0.2	12.73535	8.32E-05	1.73E-09	4.62E-14
14	0.1	0.4	13.78654	0.007234	1.83E-06	3.82E-10
14	0.25	0.4	13.21219	0.004876	1.08E-07	1.94E-09
14	0.4	0.4	12.66177	0.002971	3.40E-06	2.87E-09
14.5	0.1	0.6	14.19261	0.051202	0.000147	2.05E-07
14.5	0.25	0.6	13.43791	0.032799	1.98E-06	1.23E-06
14.5	0.4	0.6	12.72334	0.024044	0.000283	1.75E-06
15	0.1	0.8	14.61093	0.202355	0.003267	1.69E-05
15	0.25	0.8	13.71847	0.123895	8.54E-05	0.000118586
15	0.4	0.8	12.88053	0.105232	0.006457	0.000163378
15.5	0.1	1	15.04001	0.582443	0.035827	0.000496635
15.5	0.25	1	14.03945	0.342464	0.00218	0.004082092
15.5	0.4	1	13.10545	0.328434	0.072593	0.005464378

16	0.1	1.2	15.47763	1.373928	0.25217	0.007632238
16	0.25	1.2	14.38978	0.778862	0.023071	0.073262929
16	0.4	1.2	13.37839	0.827704	0.522122	0.095664048

Table 6.3 represents the absolute errors obtained against exact Crude oil future prices with $\sigma_1 = 0.358, \sigma_2 = 0.426, \rho_1 = 0.922, r = 0.06, k = 1.488, \alpha = 0.180, \tilde{\lambda} = 0.291$ for various iterations

x	y	t	Exact solution	Absolute error for n=4	Absolute error for n=10	Absolute error for n=15
16.8	0.1	0	16.8	0	0	0
16.8	0.25	0	16.8	0	0	0
16.8	0.4	0	16.8	0	0	0
17	0.1	0.2	16.87743	1.18E-04	1.37E-10	0.00E+00
17	0.25	0.2	16.44513	5.22E-05	7.08E-11	3.55E-15
17	0.4	0.2	16.02391	0.00016	4.27E-10	0
17.2	0.1	0.4	16.9837	0.003455	2.56E-07	3.36E-12
17.2	0.25	0.4	16.23286	0.001373	1.49E-07	1.50E-10
17.2	0.4	0.4	15.51521	0.00493	8.14E-07	8.54E-11
17.4	0.1	0.6	17.12047	0.02413	2.02E-05	3.39E-10
17.4	0.25	0.6	16.1311	0.008663	1.30E-05	9.30E-08
17.4	0.4	0.6	15.19891	0.036017	6.58E-05	4.88E-08
17.6	0.1	0.8	17.28519	0.09407	0.000442	1.10E-07
17.6	0.25	0.8	16.11417	0.030622	0.000308	8.77E-06
17.6	0.4	0.8	15.02248	0.145945	0.001463	4.27E-06
17.8	0.1	1	17.47394	0.267003	0.004768	7.84E-06
17.8	0.25	1	16.1621	0.07901	0.00356	0.000295266
17.8	0.4	1	14.94874	0.42842	0.016068	0.000134327
18	0.1	1.2	17.68265	0.620951	0.033017	0.000201974
18	0.25	1.2	16.25958	0.167356	0.026139	0.005191716

18	0.4	1.2	14.95104	1.026313	0.113049	0.002216134
18.2	0.1	1.4	17.90766	1.25998	0.16853	0.002906543
18.2	0.25	1.4	16.39498	0.309708	0.140351	0.058334256
18.2	0.4	1.4	15.01009	2.13805	0.585456	0.02345857
18.4	0.1	1.6	18.14588	2.315582	0.68866	0.028141152
18.4	0.25	1.6	16.55949	0.519543	0.599521	0.47252837
18.4	0.4	1.6	15.11179	4.022954	2.424333	0.179659438

Table 6.4 represents the absolute errors obtained against exact Copper future prices with $\sigma_1 = 0.274, \sigma_2 = 0.280, \rho_1 = 0.818,\ r = 0.06, k = 1.156,\ \alpha = 0.248, \tilde{\lambda} = 0.256$ for various iterations

x	y	t	Exact solution	Absolute error for n=4	Absolute error for n=10	Absolute error for n=15
110	0.1	0	110	0	0	0
110	0.25	0	110	0	0	0
110	0.4	0	110	0	0	0
115	0.1	0.2	114.1406	0.000214	4.87E-11	0
115	0.25	0.2	111.124	1.82E-05	7.82E-11	0
115	0.4	0.2	108.1871	0.000786	1.83E-10	1.42E-14
120	0.1	0.4	118.3468	0.006579	9.53E-08	7.53E-13
120	0.25	0.4	112.7958	0.000876	1.60E-07	1.78E-11
120	0.4	0.4	107.5052	0.024573	3.58E-07	3.16E-11
125	0.1	0.6	122.6398	0.048142	7.90E-06	6.47E-10
125	0.25	0.6	114.9323	0.008495	1.38E-05	1.13E-08
125	0.4	0.6	107.7092	0.182648	2.96E-05	2.07E-08
130	0.1	0.8	127.027	0.195938	0.00018	7.73E-08
130	0.25	0.8	117.4609	0.04225	0.000326	1.10E-06
130	0.4	0.8	108.6151	0.754366	0.000674	2.07E-06
135	0.1	1	131.5079	0.578772	0.002015	3.14E-06

135	0.25	1	120.3193	0.145448	0.003785	3.81E-05
135	0.4	1	110.0827	2.259281	0.007565	7.33E-05
140	0.1	1.2	136.0776	1.396819	0.014463	6.42E-05
140	0.25	1.2	123.4551	0.396649	0.027987	0.000688
140	0.4	1.2	112.0035	5.523943	0.054327	0.00135
145	0.1	1.4	140.7293	2.933822	0.076318	0.00082
145	0.25	1.4	126.824	0.921266	0.151758	0.007922
145	0.4	1.4	114.2927	11.74509	0.286925	0.015856
150	0.1	1.6	145.4553	5.56839	0.321667	0.00742
150	0.25	1.6	130.389	1.903145	0.65578	0.065729
150	0.4	1.6	116.8832	22.55041	1.210694	0.133812
155	0.1	1.8	150.2482	9.784734	1.14234	0.051578
155	0.25	1.8	134.1192	3.596065	2.382844	0.424455
155	0.4	1.8	119.7215	40.05759	4.305026	0.877632

Table 6.5 Percentage errors obtained using the equation (6.14b)

(x, y, t)		% errors for n=4	% errors for n=10
	Exact solution	in ADM	in ADM
(.1, .1, .1)	0.10851	0.00022553	3.1435e-11
(.2, .3, .4)	0.244	0.094825	0.0001258
(.3, .2, .5)	0.39039	0.35768	0.001324
(.4, .5, .6)	0.48407	0.013347	0.0051027
(.5, .4, .2)	0.55253	0.002357	6.6113e-8

Table 6.6 Percentage errors obtained using the equation (6.14b)

x	y	t	Exact	% error for 4 iterations	% error for 10 iterations	% error for 15 iterations
0.1	0.1	0.1	0.1085146503	2.25525E-04	3.14350E-11	0.00000E+00
0.1	0.3	0.5	0.1260801838	2.56160E-01	1.33993E-03	8.73688E-06
0.1	0.4	0.3	0.1148314993	1.50609E-02	5.10891E-06	2.26953E-09

0.1	0.7	0.5	0.1111070302	2.41625E-01	4.45142E-04	7.74112E-06
0.2	0.1	0.3	0.2457442266	4.25524E-02	4.68422E-06	2.47756E-09
0.2	0.3	0.4	0.2439977694	9.48254E-02	1.25804E-04	2.69613E-07
0.2	0.6	0.5	0.2293495321	1.13887E-01	2.15090E-04	2.49966E-06
0.2	0.7	0.1	0.2055422474	8.62698E-05	7.56201E-12	0.00000E+00
0.2	0.7	0.2	0.2103489536	2.70926E-03	1.69320E-08	3.80016E-12
0.2	0.7	0.3	0.2146341083	2.00488E-02	1.54463E-06	2.38308E-09
0.2	0.7	0.4	0.2185521975	8.19283E-02	3.76901E-05	2.27652E-07
0.2	0.7	0.5	0.2222140603	2.41625E-01	4.45142E-04	7.74112E-06
0.3	0.2	0.1	0.3226067264	1.87715E-04	3.92493E-11	0.00000E+00
0.3	0.4	0.2	0.3315189381	2.35700E-03	6.61128E-08	3.90147E-12
0.4	0.6	0.5	0.4586990642	1.13887E-01	2.15090E-04	2.49966E-06
0.5	0.4	0.3	0.5741574966	1.50609E-02	5.10891E-06	2.26953E-09
0.5	0.5	0.5	0.5917853240	1.57035E-02	7.65528E-04	2.53550E-06
0.6	0.5	0.4	0.6927743054	9.48283E-03	7.37756E-05	8.37084E-08
0.7	0.6	0.4	0.7862866311	3.65460E-02	2.31700E-05	6.86220E-08
0.7	0.9	0.5	0.7301077059	4.59439E-01	1.75147E-03	1.42179E-05
0.7	1	0.5	0.7073927568	5.29681E-01	2.17183E-03	1.30498E-05
0.8	0.6	0.3	0.8781246341	8.32252E-03	1.22433E-06	6.58036E-10
0.8	1	0.2	0.8007991794	6.55754E-03	1.08881E-07	7.02902E-12
0.9	0.8	0.5	0.9688526359	3.59847E-01	1.13510E-03	1.20606E-05
1	0.3	0.5	1.2608018383	2.56160E-01	1.33993E-03	8.73688E-06
1	0.7	0.4	1.0927609876	8.19283E-02	3.76901E-05	2.27652E-07
1	1	0.5	1.0105610812	5.29681E-01	2.17183E-03	1.30498E-05

Table 6.7 represents the time elapsed in seconds for finding the solution of Two Factor model with the parameter values used in table 6.1

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	1.982743	3.852880	6.545958	11.591868
VIM	1.553360	3.900412	6.441140	28.909459

Table 6.8 represents the time elapsed in seconds for finding the solution of Two Factor model with the parameter values used in table 6.2

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	1.994683	3.685885	6.651909	12.028979
VIM	1.486266	3.328135	6.485965	45.797084

Table 6.9 represents the time elapsed in seconds for finding the solution of Two Factor model with the parameter values used in table 6.3

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.005927	2.690094	6.817965	11.798120
VIM	1.472098	3.557801	5.842722	43.434506

Table 6.10 represents the time elapsed in seconds for finding the solution of Two Factor model with the parameter values used in table 6.4

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.007766	3.963014	6.477581	11.570737
VIM	1.493533	3.451490	5.960028	45.370510

Table 6.11 represents the possible values of convergence control parameter (c_0) obtained for $\sigma_1 = 0.393, \sigma_2 = 0.527, \rho_1 = 0.766, r = 0.06, k = 1.876, \alpha = 0.106, \tilde{\lambda} = 0.198$ using HAM for n=4

$x = 20, y = .10, t = 0.2$	$x = 20.5, y = .10, t = 0.4$
15.4971763699993 + 0.00000000000000i	7.74858818499964 + 0.00000000000000i

-9.12228036274570 + 0.00000000000000i	-4.56114018137285 + 0.00000000000000i
2.73116035551266 + 7.96201323605130i	1.36558017775633 + 3.98100661802565i
2.73116035551266 - 7.96201323605130i	1.36558017775633 - 3.98100661802565i
$x = 21, y = .10, t = 0.6$	$x = 21.5, y = .10, t = 0.8$
5.16572545666643 + 0.00000000000000i	3.87429409249982 + 0.00000000000000i
-3.04076012091523 + 0.00000000000000i	-2.28057009068643 + 0.00000000000000i
0.910386785170886 + 2.65400441201710i	0.682790088878165 + 1.99050330901283i
0.910386785170886 - 2.65400441201710i	0.682790088878165 - 1.99050330901283i
$x = 22, y = .10, t = 1$	$x = 22.5, y = .10, t = 1.2$
3.09943527399986 + 0.00000000000000i	2.58286272833321 + 0.00000000000000i
-1.82445607254914 + 0.00000000000000i	-1.52038006045762 + 0.00000000000000i
0.546232071102532 + 1.59240264721026i	0.455193392585444 + 1.32700220600855i
0.546232071102532 - 1.59240264721026i	0.455193392585444 - 1.32700220600855i
$x = 23, y = .10, t = 1.4$	$x = 23.5, y = .10, t = 1.6$
2.21388233857133 + 0.00000000000000i	1.93714704624991 + 0.00000000000000i
-1.30318290896367 + 0.00000000000000i	-1.14028504534321 + 0.00000000000000i

0.390165765073237 + 1.13743046229304i	0.341395044439083 + 0.995251654506413i
0.390165765073237 - 1.13743046229304i	0.341395044439083 - 0.995251654506413i
$x = 20, y = .15, t = 0.2$	$x = 20.5, y = .15, t = 0.4$
330.401423735203 + 0.00000000000000i	165.200711867602 + 0.00000000000000i
14.3787946370860 + 0.00000000000000i	7.18939731854297 + 0.00000000000000i
1.11878151904653 + 5.18843070902909i	0.559390759523271 + 2.59421535451455i
1.11878151904653 - 5.18843070902909i	0.559390759523271 - 2.59421535451455i
$x = 21, y = .15, t = 0.6$	$x = 21.5, y = .15, t = 0.8$
110.133807911734 + 0.00000000000000i	82.6003559338006 + 0.00000000000000i
4.79293154569532 + 0.00000000000000i	3.59469865927149 + 0.00000000000000i
0.372927173015514 + 1.72947690300970i	0.279695379761635 + 1.29710767725727i
0.372927173015514 - 1.72947690300970i	0.279695379761635 - 1.29710767725727i
$x = 22, y = .15, t = 1$	$x = 22.5, y = .15, t = 1.2$
66.0802847470406 + 0.00000000000000i	55.0669039558673 + 0.00000000000000i
2.87575892741719 + 0.00000000000000i	2.39646577284766 + 0.00000000000000i
0.223756303809308 + 1.03768614180582i	0.186463586507756 + 0.864738451504848i

0.223756303809308 - 1.03768614180582i	0.186463586507756 - 0.864738451504848i
$x = 23, y = .15, t = 1.4$	$x = 23.5, y = .15, t = 1.6$
47.2002033907433 + 0.00000000000000i	41.3001779669004 + 0.00000000000000i
2.05411351958371 + 0.00000000000000i	1.79734932963574 + 0.00000000000000i
0.159825931292362 + 0.741204387004156i	0.139847689880817 + 0.648553838628637i
0.159825931292362 - 0.741204387004156i	0.139847689880817 - 0.648553838628637i

6.7.2 DISCUSSION

This study has undertaken parameters as were available against pre-executed study by Schwartz [111] for crude oil and copper, say, speed adjustment of spot commodity prices (k), volatility of spot price of commodity (σ_1), volatility of convenience yield (σ_2), risk free interest rate (r), log run mean price of the convenience yield (α), market price of risk ($\tilde{\lambda}$), correlation between spot price and convenience yield of a commodity (ρ_1) and were substituted in the two-factor model, and subsequently, crude oil and copper future prices were found.

Prior to the obtained prices, on substitution of parameters, the achieved polynomial (A 2.1.1) and the corresponding coefficients were tested successfully for convergence (A 2.3.1 & A 2.3.2), and thus approximate solution was derived. On comparison with exact solutions derived from analytical solution of the existing study, the obtained approximate solution values from both ADM and VIM methods are found to be accurately identical. The obtained errors (tables 6.1 to 6.3) are found to be reduced significantly with increased number of iterations.

On the similar lines, present study as refereed above, commodity prices for copper were subjected to two-factor model, and this study resultant approximate solution (A 2.1.2) using ADM and VIM methods, and as a matter of successful application the obtained values are precisely matching with exact solutions, and the errors (tables 6.4 to 6.6), were significantly reduced with increased number of iterations.

Further, the study has been examined to accessible time lapse between the ADM and VIM methods, for both crude oil and copper prices. The observed delay for 10 iterations or n=10, is largely similar with both the methods, but further increase in the number of iterations, the delay mostly in case of VIM found to be approximately 6 times. Detailed delay specific tables were shown in tables 6.7 to 6.10.

The study further estimated for HAM method on the above refereed study, and thus obtained polynomial in terms of four variables, that are, x, y, t, and c_0 (6.15). When the parameters of the refereed study, $x, y,$ and t were substituted in the above polynomial, 8-sets of polynomials were derived (A 2.2.1 to A 2.2.8). However, the study observed that the c_0 value happens to be not constant among all the obtained 8-sets of polynomials (table 6.11), an impediment while testing validity of HAM.

6.8 CONCLUSION

The validation of ADM, VIM, HAM and HPM methods, were carry out for originating approximate solutions of two-factor commodity price model, in the form of polynomials which in turn will be of enormous help while efficaciously calculating the future commodity prices at any short interval of time more accurately and with less degree of error. The obtained errors are to be reduced significantly with increased number of iterations. However, computations through VIM are of extensive duration compared to ADM to obtain the approximate polynomial. Lack of convergence control parameter c_0 is one of the significant limitations while solving the two-factor commodity price model using HAM and HPM.

CHAPTER-7

THREE FACTOR COMMODITY PRICE MODEL AND ITS SOLUTION

7.1 THREE FACTOR COMMODITY PRICE MODEL

The Three Factor Commodity Price Model is

$$\frac{\sigma_1^2}{2}x^2u_{xx} + \frac{\sigma_2^2}{2}u_{yy} + \frac{\sigma_3^2}{2}u_{zz} + \sigma_1\sigma_2\rho_1 xu_{xy} + \sigma_2\sigma_3\rho_2 u_{yz} + \sigma_1\sigma_3\rho_3 xu_{xz} +$$
$$(z-y)xu_x + k(\hat{\alpha}-y)u_y + a(m^*-z)u_z = u_t \tag{7.1}$$

with the terminal boundary condition $u(x,y,z,0) = x$ (7.2)

The closed form solution of the equation (7.1) along with (7.2) found to be

$$u(x,y,z,t) = x\exp\left[C(t) - y\frac{1-e^{-kt}}{k} + z\frac{1-e^{-at}}{a}\right] \tag{7.3}$$

where

$C(t) =$

$$\left[\begin{array}{c} \left\{\frac{(1-kt-e^{-kt})\,(k\hat{\alpha}+\sigma_1\sigma_2\rho_1)}{k^2}\right\} - \frac{\sigma_2^2\{4(1-e^{-kt})-(1-e^{-2kt})-2kt\}}{4k^3} \\ -\frac{\{(1-at-e^{-at})\,(am^*+\sigma_1\sigma_3\rho_3)\}}{a^2} - \frac{\sigma_3^2\{4(1-e^{-at})-(1-e^{-2at})-2at\}}{4a^3} \\ +\sigma_2\sigma_3\rho_2\left\{\frac{\left((1-e^{-kt})+(1-e^{-at})-(1-e^{-(k+a)t})\right)}{ka(k+a)} + \frac{k^2(1-e^{-at})+a^2(1-e^{-at})-ka^2t-ak^2t}{k^2a^2(k+a)}\right\} \end{array}\right]$$

(7.4)

7.2 SOLUTION OF THREE FACTOR COMMODITY PRICE MODEL USING ADM

According to ADM, approximate solution of the three factor commodity price model equation can be written as

$$u = u(x,y,z,0) + L^{-1}\Big[\frac{\sigma_1^2}{2}x^2u_{xx} + \frac{\sigma_2^2}{2}u_{yy} + \frac{\sigma_3^2}{2}u_{zz} + \sigma_1\sigma_2\rho_1 x u_{xy} + \sigma_2\sigma_3\rho_2 u_{yz} + \sigma_1\sigma_3\rho_3 x u_{xz} + (z-y)xu_x + k(\hat{\alpha}-y)u_y + a(m^*-z)u_z\Big] \quad (7.5)$$

$$u_0 = x$$

$$u_{n+1} = \int_0^t A_n\,dt \ \text{ for } n \geq 1$$

7.3 SOLUTION OF THREE FACTOR COMMODITY PRICE MODEL USING VIM

The approximate solution of the Three Factor model equation, using VIM, can be written as follows

$$u_{n+1} = u_n(x,y,z,t) + \int_0^t \phi\Big[\frac{\partial u_n(x,y,z,t)}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 u_n}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 u_n}{\partial y\,\partial z} - \sigma_1\sigma_3\rho_3 x\frac{\partial^2 u_n}{\partial x\,\partial z} - (z-y)x\frac{\partial u_n}{\partial x} - k(\hat{\alpha}-y)\frac{\partial u_n}{\partial y} - a(m^*-z)\frac{\partial u_n}{\partial z}\Big]dt \quad (7.6)$$

$$\delta u_{n+1} = \delta u_n(x,y,z,t) + \delta\int_0^t \phi\Big[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 \widetilde{u_n}}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 \widetilde{u_n}}{\partial y\,\partial z} - \sigma_1\sigma_3\rho_3 x\frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial z} - (z-y)x\frac{\partial \widetilde{u_n}}{\partial x} - k(\hat{\alpha}-y)\frac{\partial \widetilde{u_n}}{\partial y} - a(m^*-z)\frac{\partial \widetilde{u_n}}{\partial z}\Big]dt \quad (7.7)$$

$$\delta u_{n+1} = \delta u_n(x,y,z,t) + \delta\int_0^t \phi\Big[\frac{\partial u_n}{\partial t}\Big]dt \quad (7.8)$$

$$\delta u_{n+1} = \delta u_n(x,y,z,t)(1+\phi) - \delta \int_0^t \phi' \delta u_n(x,y,z,t)dt \tag{7.9}$$

This yields stationary conditions

$$1+\phi = 0,\ \phi' = 0 \Longrightarrow \phi = -1 \tag{7.10}$$

Substituting the value of $\phi = -1$ into the functional (7.6) give the iteration formulas

$$u_{n+1} = u_n(x,y,z,t) - \int_0^t \left[\frac{\partial u_n}{\partial t} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 u_n}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 u_n}{\partial y\,\partial z} - \sigma_1\sigma_3\rho_3 x\frac{\partial^2 u_n}{\partial x\,\partial z} - (z-y)x\frac{\partial u_n}{\partial x} - k(\hat{\alpha}-y)\frac{\partial u_n}{\partial y} - a(m^* - z)\frac{\partial u_n}{\partial z}\right]dt \tag{7.11}$$

7.4 CONVERGENCE OF SOLUTION OF THREE FACTOR COMMODITY PRICE MODEL

7.4.1 Convergence of solution of Three Factor Commodity Price Model using VIM has been verified as given in the following theorem developed in our investigation.

(Pannala & et al. [103])

Equation (7.1), can be re-written as

$$u_t - c_{31}x^2u_{xx} - c_{32}u_{yy} - c_{33}u_{zz} - c_{34}xu_{xy} - c_{35}u_{yz} - c_{36}xu_{xz} - c_{37}u_y - c_{38}u_z + c_{39}yu_y + c_{310}zu_z - (z-y)xu_x = 0 \tag{7.12}$$

where $c_{31} = \frac{\sigma_1^2}{2}, c_{32} = \frac{\sigma_2^2}{2}, c_{33} = \frac{\sigma_3^2}{2}, c_{34} = \sigma_1\sigma_2\rho_1, c_{35} = \sigma_2\sigma_3\rho_2, c_{36} = \sigma_1\sigma_3\rho_3,$

$c_{37} = k\hat{\alpha}, c_{38} = am^*, c_{39} = k, c_{310} = a$

Let us consider the functions, $F_{31} \equiv x^2u_{xx},\ F_{32} \equiv u_{yy},\ F_{33} \equiv u_{zz},$

$F_{34} \equiv xu_{xy},\ F_{35} \equiv u_{yz},\ F_{36} \equiv xu_{xz},\ F_{37} \equiv u_y,\ F_{38} \equiv u_z, F_{39} \equiv yu_y,$

$F_{310} \equiv zu_z, F_{311} \equiv (z-y)xu_x$ are Lipschitz continuous with

$|F_{3i}(u) - F_{3i}(u^*)| \leq L_{3i}|u - u^*|$ for $x, y, z > 0$, and $J = [0, T]$ $(T \in \mathbb{R})$ (7.13)

where $i = 1$ to 11 and $c_{311} = 1$ (7.14)

Theorem 7.1: The solution $u_n(x, y, z, t)$ obtained from (7.11) converges to the solution of problem (7.1) when $0 < \beta_{31} < 1$ and $0 < \beta_{32} < 1$ where

$\beta_{31} = \sum_{j=1}^{11} |c_{3j}| \, L_{3j}$ and $\beta_{32} = [1 - T(1 - \beta_{31})]$

Proof: Consider,

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \int_0^t \left[\frac{\partial u_n}{\partial t} - c_{31}x^2 \frac{\partial^2 u_n}{\partial x^2} - c_{32} \frac{\partial^2 u_n}{\partial y^2} - c_{33} \frac{\partial^2 u_n}{\partial z^2} - c_{34}x \frac{\partial^2 u_n}{\partial x \, \partial y} - c_{35} \frac{\partial^2 u_n}{\partial y \, \partial z} - c_{36}x \frac{\partial^2 u_n}{\partial x \, \partial z} - c_{37} \frac{\partial u_n}{\partial y} - c_{38} \frac{\partial u_n}{\partial z} + c_{39}y \frac{\partial u_n}{\partial y} + c_{310}z \frac{\partial u_n}{\partial z} - (z - y)x \frac{\partial u_n}{\partial x}\right] dt$$

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial t} - \sum_{j=1}^{8} c_{3j} \, F_{3j}(u_n) + c_{39}F_{39}(u_n) + c_{310}F_{310}(u_n) - F_{311}(u_n)\right] dt \tag{7.15}$$

Let us consider $u_n = u - \int_0^t \left[\frac{\partial u}{\partial t} - \sum_{j=1}^{8} c_{3j} \, F_{3j}(u) + c_{39}F_{39}(u) + c_{310}F_{310}(u) - F_{311}(u)\right] dt$ (7.16)

Let $e_{n+1}(x, y, z, t) = u_{n+1} - u_n$, $e_n(x, y, z, t) = u_n - u$

$|e_n(x, y, z, t^*)| = \max_t |e_n(x, y, z, t)|$. Since e_n is a decreasing function with respect to 't' then from mean value theorem and (7.15)-(7.16), we obtained,

$$e_{n+1} = e_n + \int_0^t \left[\frac{\partial(-e_n)}{\partial t} + \sum_{j=1}^{8} c_{3j} \, \{F_{3j}(u_n) - F_{3j}(u)\} - c_{39}\{F_{39}(u_n) - F_{39}(u)\} - c_{310}\{F_{310}(u_n) - F_{310}(u) \} + \{F_{311}(u_n) - F_{311}(u)\}\right] dt$$

$$e_{n+1} \leq e_n + \int_0^t (-e_n) dt + \left[\sum_{j=1}^{11} |c_{3j}| \, L_{3j}\right] \int_0^t |e_n| dt$$

$$e_{n+1} \leq e_n - Te_n(x, y, z, \omega) + \left[\sum_{j=1}^{11} |c_{3j}| \, L_{3j}\right] T|e_n|$$

$e_{n+1}(x, y, z, t) \leq [1 - T\{1 - \beta_{31}\}]|e_n(x, y, z, t^*)|$ where $0 \leq \omega \leq t$

Hence, $e_{n+1}(x, y, z, t) \leq \beta_{32}|e_n(x, y, z, t^*)|$, therefore,

$\|e_{n+1}\| = max_{\forall t \in J}|e_{n+1}| \leq \beta_{32}\, max_{\forall t \in J}|e_n| \leq \beta_{32}\|e_n\|$

Since, $0 < \beta_{32} < 1$, then $\|e_n\| \to 0$

7.4.2 Convergence of solution of Three Factor Commodity Price Model using ADM

It was discussed in section 5.4.2

7.5 NUMERICAL EXAMPLES

Example 7.1: Consider $\sigma_1 = \sigma_2 = \sigma_3 = a = m^* = 1 = \rho_1 = \rho_2 = \rho_3$ and $k = \tilde{\lambda} = \alpha = 2$ in the equation (7.1),

We obtain the following approximants using ADM for n=3

$u_0 = x$

$u_1 = -t * x * (y - z)$

$u_2 = t^2 * \left(\frac{x*(2*y-2)}{2} - \frac{x*(z-1)}{2} + \frac{x*(y-z)^2}{2}\right)$

$$u_3 = -\frac{t^3*x*\left(y^3 - 3*y^2*z + 6*y^2 + 3*y*z^2 - 9*y*z + y - z^3 + 3*z^2 + 2*z - 4\right)}{6} \qquad (7.17a)$$

Adding all the approximants in (7.17a) we obtain the approximate solution of (7.1) for n=3, as

$$u(x, y, z, t) = x - t * x * (y - z) + t^2 * \left(\frac{x*(2*y-2)}{2} - \frac{x*(z-1)}{2} + \frac{x*(y-z)^2}{2}\right) - \frac{t^3*x*\left(y^3 - 3*y^2*z + 6*y^2 + 3*y*z^2 - 9*y*z + y - z^3 + 3*z^2 + 2*z - 4\right)}{6} \qquad (7.17b)$$

Example 7.2: Solved the example-7.1 using VIM

We obtain the following approximant for three iterations using VIM

$u_3 =$

$\left(x - xt(y - z) + \frac{xt^2(y^2 - 2yz + 2y + z^2 - z - 1)}{2!} - \right.$

$\left. \frac{x\, t^3(y - z + 1)(y^2 - 2yz + 5y + z^2 - 2z - 4)}{3!} \right)$

7.6 SOLUTION OF THREE FACTOR COMMODITY PRICE MODEL USING HAM, AND HPM

7.6.1 Solution using HAM:

Using a set of parameter values of Crude oil $\sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = .0081, \rho_1 = 0.915, \rho_2 = -.0039, \rho_3 = -.0293, r = 0.06, k = 1.314, \alpha = 0.249, \tilde{\lambda} = 0.353, a = .2, m^* = 0.07082$, obtained the following polynomial in x, y, z, t, c_0 with the help of MATLAB for n=4,

$u_4 = x * \big(c_0^2 * t * (y - 1.0 * z) - 0.2502 * c_0^3 * t^2 - 0.0926 * c_0^3 * t^3 -$

$0.09381 * c_0^4 * t^2 - 0.06945 * c_0^4 * t^3 - 0.01104 * c_0^4 * t^4 - 0.5 * c_0 * t^2 *$

$\big(0.05307 * c_0 - 1.0 * c_0 * (y - 1.0 * z)^2 - 1.0 * c_0 * (1.045 * y -$

$0.02347) + c_0 * (0.2 * z - 0.014)\big) - 0.1564 * c_0^2 * t^2 + 5.0 * c_0^2 * t * y +$

$4.0 * c_0^3 * t * y + c_0^4 * t * y - 5.0 * c_0^2 * t * z - 4.0 * c_0^3 * t * z - 1.0 * c_0^4 *$

$t * z + 2.613 * c_0^2 * t^2 * y + 4.18 * c_0^3 * t^2 * y + 0.6029 * c_0^3 * t^3 * y +$

$1.568 * c_0^4 * t^2 * y + 0.4522 * c_0^4 * t^3 * y + 0.008061 * c_0^4 * t^4 * y - 0.5 *$

$c_0^2 * t^2 * z - 0.8 * c_0^3 * t^2 * z + 0.09842 * c_0^3 * t^3 * z - 0.3 * c_0^4 * t^2 * z +$

$0.07381 * c_0^4 * t^3 * z + 0.02594 * c_0^4 * t^4 * z + 2.0 * c_0 * t * (y - 1.0 * z) +$

$2.0 * c_0 * t * y - 2.0 * c_0 * t * z + 2.5 * c_0^2 * t^2 * y^2 + 4.0 * c_0^3 * t^2 * y^2 +$

$2.09 * c_0^3 * t^3 * y^2 + 1.5 * c_0^4 * t^2 * y^2 + 0.6667 * c_0^3 * t^3 * y^3 + 1.568 * c_0^4 *$

$t^3 * y^2 + 0.5 * c_0^4 * t^3 * y^3 + 0.3029 * c_0^4 * t^4 * y^2 + 0.2612 * c_0^4 * t^4 * y^3 +$

$0.04167 * c_0^4 * t^4 * y^4 + 2.5 * c_0^2 * t^2 * z^2 + 4.0 * c_0^3 * t^2 * z^2 + 0.4 * c_0^3 * t^3 * z^2 + 1.5 * c_0^4 * t^2 * z^2 - 0.6667 * c_0^3 * t^3 * z^3 + 0.3 * c_0^4 * t^3 * z^2 - 0.5 * c_0^4 * t^3 * z^3 - 0.003969 * c_0^4 * t^4 * z^2 - 0.05 * c_0^4 * t^4 * z^3 + 0.04167 * c_0^4 * t^4 * z^4 + 2.0 * c_0^3 * t^3 * y * z^2 - 2.0 * c_0^3 * t^3 * y^2 * z + 1.5 * c_0^4 * t^3 * y * z^2 - 1.5 * c_0^4 * t^3 * y^2 * z + 0.3612 * c_0^4 * t^4 * y * z^2 - 0.5725 * c_0^4 * t^4 * y^2 * z - 0.1667 * c_0^4 * t^4 * y * z^3 - 0.1667 * c_0^4 * t^4 * y^3 * z + 0.25 * c_0^4 * t^4 * y^2 * z^2 - 5.0 * c_0^2 * t^2 * y * z - 8.0 * c_0^3 * t^2 * y * z - 2.49 * c_0^3 * t^3 * y * z - 3.0 * c_0^4 * t^2 * y * z - 1.868 * c_0^4 * t^3 * y * z - 0.2096 * c_0^4 * t^4 * y * z + 1.0)$ (7.18)

where c_0 is called convergence control parameter.

7.6.2 Solution using HPM

If $c_0 = -1$, then HAM will be in the form of HPM.

7.7 RESULTS AND DISCUSSION

7.7.1 RESULTS

The risk-adjusted drift of the process:

$m^* = R(\infty) + \frac{\sigma_3^2}{2\,a^2} = 0.07 + \left(\frac{0.0081^2}{2\,(0.2^2)}\right) = 0.07082$ for Crude Oil (Schwartz, 1997 page no: 949)

$m^* = R(\infty) + \frac{\sigma_3^2}{2\,a^2} = 0.07 + \left(\frac{0.0096^2}{2\,(0.2^2)}\right) = 0.071152$ for Copper (Schwartz, 1997 page no: 949)

where $R(\infty)$ is called infinite maturity discount yield be 7 percent

The following tables 7.1 to 7.5 have prepared using ADM and VIM

Table 7.1 represents the absolute errors obtained against exact Crude oil future prices with $k = 1.314, \alpha = 0.249, \sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = 0.0081, \rho_1 = 0.915, \rho_2 = -0.0039,\ \rho_3 = -0.0293, \tilde{\lambda} = 0.353, a = 0.2, m^* = 0.07082$ for various iterations

x	y	z	t	Exact solution	Absolute error for n=4	Absolute error for n=10	Absolute error for n=15
19.5	0.1	0.03	0	19.5	0	0	0
19.5	0.25	0.03	0	19.5	0	0	0
19.5	0.4	0.03	0	19.5	0	0	0
19.5	0.1	0.06	0	19.5	0	0	0
19.5	0.1	0.09	0	19.5	0	0	0
19.5	0.25	0.09	0	19.5	0	0	0
19.5	0.25	0.06	0	19.5	0	0	0
19.5	0.4	0.06	0	19.5	0	0	0
19.5	0.4	0.09	0	19.5	0	0	0
20	0.1	0.03	0.2	19.7426	8.63E-05	5.12E-11	3.55E-15
20	0.25	0.03	0.2	19.2286	2.16E-05	4.60E-11	0
20	0.4	0.03	0.2	18.728	0.00018	1.92E-10	3.55E-15
20	0.1	0.06	0.2	19.8591	8.28E-05	4.95E-11	7.11E-15
20	0.1	0.09	0.2	19.9762	7.92E-05	4.77E-11	0
20	0.25	0.09	0.2	19.4561	2.77E-05	3.38E-11	3.55E-15
20	0.25	0.06	0.2	19.342	2.49E-05	3.96E-11	3.55E-15
20	0.4	0.06	0.2	18.8384	0.00016	1.80E-10	3.55E-15
20	0.4	0.09	0.2	18.9496	0.00015	1.68E-10	3.55E-15
20.5	0.1	0.03	0.4	20.021	0.00257	9.71E-08	7.64E-13
20.5	0.25	0.03	0.4	19.1082	0.00052	9.47E-08	3.79E-11
20.5	0.4	0.03	0.4	18.237	0.00549	3.70E-07	2.42E-12
20.5	0.1	0.06	0.4	20.2533	0.00247	9.41E-08	2.56E-13
20.5	0.1	0.09	0.4	20.4882	0.00237	9.09E-08	1.14E-12
20.5	0.25	0.09	0.4	19.554	0.00073	7.04E-08	3.27E-11
20.5	0.25	0.06	0.4	19.3298	0.00063	8.20E-08	3.52E-11
20.5	0.4	0.06	0.4	18.4485	0.005	3.47E-07	4.75E-12
20.5	0.4	0.09	0.4	18.6625	0.00454	3.25E-07	6.74E-12
21	0.1	0.03	0.6	20.337	0.0183	7.83E-06	8.65E-10

21	0.25	0.03	0.6	19.1094	0.00295	8.17E-06	2.38E-08
21	0.4	0.03	0.6	17.9559	0.04032	3.03E-05	4.97E-10
21	0.1	0.06	0.6	20.6849	0.01761	7.60E-06	1.99E-10
21	0.1	0.09	0.6	21.0388	0.01692	7.36E-06	3.80E-10
21	0.25	0.09	0.6	19.7688	0.00452	6.14E-06	2.05E-08
21	0.25	0.06	0.6	19.4363	0.00379	7.11E-06	2.21E-08
21	0.4	0.06	0.6	18.263	0.03678	2.85E-05	2.04E-09
21	0.4	0.09	0.6	18.5754	0.03348	2.67E-05	3.35E-09
21.5	0.1	0.03	0.8	20.6888	0.07252	0.00017	1.15E-07
21.5	0.25	0.03	0.8	19.2082	0.00898	0.00019	2.27E-06
21.5	0.4	0.03	0.8	17.8336	0.16458	0.00068	3.76E-08
21.5	0.1	0.06	0.8	21.1528	0.06991	0.00017	4.93E-08
21.5	0.1	0.09	0.8	21.6272	0.06728	0.00016	8.25E-09
21.5	0.25	0.09	0.8	20.0794	0.01567	0.00015	1.97E-06
21.5	0.25	0.06	0.8	19.639	0.01254	0.00017	2.12E-06
21.5	0.4	0.06	0.8	18.2335	0.15028	0.00064	1.16E-07
21.5	0.4	0.09	0.8	18.6424	0.13695	0.0006	2.48E-07
22	0.1	0.03	1	21.0731	0.20908	0.0019	4.92E-06
22	0.25	0.03	1	19.3854	0.0189	0.00222	7.76E-05
22	0.4	0.03	1	17.8329	0.48705	0.0076	3.84E-06
22	0.1	0.06	1	21.654	0.20184	0.00185	2.60E-06
22	0.1	0.09	1	22.2508	0.1945	0.0018	5.70E-07
22	0.25	0.09	1	20.4688	0.03935	0.00169	6.72E-05
22	0.25	0.06	1	19.9198	0.02977	0.00194	7.23E-05
22	0.4	0.06	1	18.3244	0.44515	0.00715	1.59E-06
22	0.4	0.09	1	18.8295	0.40608	0.00671	6.26E-06
22.5	0.1	0.03	1.2	21.4863	0.49342	0.01335	0.0001
22.5	0.25	0.03	1.2	19.6259	0.02971	0.0163	0.00138
22.5	0.4	0.03	1.2	17.9265	1.17686	0.05412	0.00011
22.5	0.1	0.06	1.2	22.1851	0.47691	0.01303	6.09E-05
22.5	0.1	0.09	1.2	22.9067	0.46013	0.01268	2.37E-05
22.5	0.25	0.09	1.2	20.9232	0.08047	0.01248	0.0012
22.5	0.25	0.06	1.2	20.2642	0.05665	0.0143	0.00129
22.5	0.4	0.06	1.2	18.5096	1.0765	0.05092	9.29E-06
22.5	0.4	0.09	1.2	19.1116	0.98286	0.04784	7.67E-05
23	0.1	0.03	1.4	21.9249	1.01513	0.06903	0.00134
23	0.4	0.03	1.4	18.0941	2.47352	0.28354	0.00165
23	0.1	0.06	1.4	22.7429	0.98226	0.0675	0.00084

23	0.1	0.09	1.4	23.5915	0.94873	0.06577	0.00041
23	0.25	0.09	1.4	21.4316	0.14238	0.06751	0.01366
23	0.25	0.06	1.4	20.6607	0.09108	0.07719	0.01467
23	0.4	0.06	1.4	18.7692	2.26418	0.26688	0.00049
23	0.4	0.09	1.4	19.4695	2.06877	0.25087	0.00052
23.5	0.1	0.03	1.6	22.3857	1.89013	0.2856	0.01213
23.5	0.4	0.03	1.6	18.3201	4.69614	1.18723	0.01657
23.5	0.1	0.06	1.6	23.3244	1.83076	0.27967	0.008
23.5	0.1	0.09	1.6	24.3024	1.77	0.27284	0.00437
23.5	0.4	0.06	1.6	19.0884	4.30136	1.11798	0.00681
23.5	0.4	0.09	1.6	19.8888	3.93266	1.05142	0.00165

Table 7.2 represents the absolute errors obtained against exact Copper future prices with $k = 1.045, \alpha = 0.255, \sigma_1 = 0.266, \sigma_2 = 0.249,\ \sigma_3 = 0.0096,\ \rho_1 = 0.805,\ \rho_2 = 0.1243,\ \rho_3 = 0.0964,\ a = 0.2,\ m^* = 0.071152,\ \tilde{\lambda} = 0.243$ for various iterations

x	y	z	t	Exact Solution	Absolute Error for n=4	Absolute Error for n=10	Absolute Error for n=15
110	0.1	0.03	0	110	0	0	0
110	0.1	0.06	0	110	0	0	0
110	0.1	0.09	0	110	0	0	0
110	0.25	0.03	0	110	0	0	0
110	0.25	0.06	0	110	0	0	0
110	0.25	0.09	0	110	0	0	0
110	0.4	0.03	0	110	0	0	0
110	0.4	0.06	0	110	0	0	0
110	0.4	0.09	0	110	0	0	0
115	0.1	0.03	0.2	113.487	0.00014	1.79E-11	0
115	0.1	0.06	0.2	114.156	0.00014	1.74E-11	1.42E-14
115	0.1	0.09	0.2	114.83	0.00013	1.68E-11	1.42E-14
115	0.25	0.03	0.2	110.456	6.85E-05	4.27E-11	0
115	0.25	0.06	0.2	111.107	5.08E-05	3.76E-11	1.42E-14
115	0.25	0.09	0.2	111.763	3.53E-05	3.30E-11	1.42E-14
115	0.4	0.03	0.2	107.506	0.00073	5.80E-11	1.42E-14
115	0.4	0.06	0.2	108.14	0.00066	5.74E-11	1.42E-14

115	0.4	0.09	0.2	108.778	0.0006	5.63E-11	2.84E-14
120	0.1	0.03	0.4	117.047	0.00442	3.50E-08	5.54E-13
120	0.1	0.06	0.4	118.405	0.00422	3.41E-08	3.41E-13
120	0.1	0.09	0.4	119.778	0.00401	3.30E-08	1.71E-13
120	0.25	0.03	0.4	111.445	0.00235	8.69E-08	4.38E-12
120	0.25	0.06	0.4	112.738	0.00179	7.66E-08	4.14E-12
120	0.25	0.09	0.4	114.046	0.00129	6.73E-08	3.88E-12
120	0.4	0.03	0.4	106.112	0.02287	1.12E-07	1.42E-11
120	0.4	0.06	0.4	107.343	0.02081	1.11E-07	1.25E-11
120	0.4	0.09	0.4	108.588	0.0189	1.09E-07	1.10E-11
125	0.1	0.03	0.6	120.698	0.03244	2.90E-06	4.11E-10
125	0.1	0.06	0.6	122.762	0.031	2.84E-06	2.72E-10
125	0.1	0.09	0.6	124.862	0.02953	2.75E-06	1.53E-10
125	0.25	0.03	0.6	112.891	0.01887	7.47E-06	2.79E-09
125	0.25	0.06	0.6	114.823	0.01456	6.60E-06	2.64E-09
125	0.25	0.09	0.6	116.787	0.01075	5.81E-06	2.48E-09
125	0.4	0.03	0.6	105.59	0.16997	9.11E-06	9.27E-09
125	0.4	0.06	0.6	107.396	0.15473	9.09E-06	8.16E-09
125	0.4	0.09	0.6	109.233	0.14066	8.99E-06	7.15E-09
130	0.1	0.03	0.8	124.446	0.13226	6.60E-05	4.39E-08
130	0.1	0.06	0.8	127.237	0.12664	6.47E-05	2.99E-08
130	0.1	0.09	0.8	130.091	0.12084	6.30E-05	1.80E-08
130	0.25	0.03	0.8	114.726	0.083	0.00018	2.70E-07
130	0.25	0.06	0.8	117.299	0.06484	0.00016	2.56E-07
130	0.25	0.09	0.8	119.93	0.04879	0.00014	2.41E-07
130	0.4	0.03	0.8	105.766	0.70205	0.0002	9.18E-07
130	0.4	0.06	0.8	108.138	0.63951	0.0002	8.08E-07
130	0.4	0.09	0.8	110.563	0.58168	0.0002	7.09E-07
135	0.1	0.03	1	128.294	0.39124	0.00074	1.65E-06
135	0.1	0.06	1	131.83	0.37527	0.00073	1.15E-06
135	0.1	0.09	1	135.464	0.35868	0.00071	7.23E-07
135	0.25	0.03	1	116.894	0.26212	0.00203	9.34E-06
135	0.25	0.06	1	120.116	0.20684	0.0018	8.87E-06
135	0.25	0.09	1	123.427	0.15787	0.00159	8.37E-06
135	0.4	0.03	1	106.507	2.10301	0.00227	3.24E-05
135	0.4	0.06	1	109.442	1.91664	0.00228	2.85E-05
135	0.4	0.09	1	112.459	1.74426	0.00227	2.51E-05
140	0.1	0.03	1.2	132.238	0.94543	0.00531	3.19E-05

140	0.1	0.06	1.2	136.539	0.90827	0.00523	2.27E-05
140	0.1	0.09	1.2	140.98	0.86945	0.00512	1.47E-05
140	0.25	0.03	1.2	119.346	0.67067	0.01498	0.00017
140	0.25	0.06	1.2	123.227	0.53351	0.01328	0.00016
140	0.25	0.09	1.2	127.235	0.41181	0.01173	0.00015
140	0.4	0.03	1.2	107.71	5.14317	0.01607	0.00059
140	0.4	0.06	1.2	111.213	4.68956	0.01622	0.00052
140	0.4	0.09	1.2	114.83	4.26977	0.01619	0.00046
145	0.1	0.03	1.4	136.274	1.98788	0.02798	0.00039
145	0.1	0.06	1.4	141.359	1.91255	0.02765	0.00028
145	0.1	0.09	1.4	146.633	1.83338	0.02709	0.00019
145	0.25	0.03	1.4	122.042	1.48316	0.0811	0.00194
145	0.25	0.06	1.4	126.595	1.18775	0.07197	0.00184
145	0.25	0.09	1.4	131.319	0.92525	0.06364	0.00174
145	0.4	0.03	1.4	109.296	10.9387	0.08389	0.00695
145	0.4	0.06	1.4	113.374	9.97806	0.08497	0.00613
145	0.4	0.09	1.4	117.604	9.08862	0.08502	0.00539
150	0.1	0.03	1.6	140.397	3.77644	0.11781	0.00342
150	0.1	0.06	1.6	146.284	3.63823	0.11671	0.00249
150	0.1	0.09	1.6	152.418	3.49215	0.11457	0.0017
150	0.25	0.03	1.6	124.948	2.94714	0.34994	0.01604
150	0.25	0.06	1.6	130.188	2.37349	0.31082	0.01528
150	0.25	0.09	1.6	135.647	1.86307	0.27508	0.01448
150	0.4	0.03	1.6	111.199	21.0087	0.34996	0.05842
150	0.4	0.06	1.6	115.862	19.1708	0.35565	0.05158
150	0.4	0.09	1.6	120.721	17.4684	0.3568	0.04542
155	0.1	0.03	1.8	144.601	6.641	0.41788	0.02317
155	0.1	0.06	1.8	151.309	6.40592	0.41491	0.01706
155	0.1	0.09	1.8	158.329	6.15607	0.40807	0.0118
155	0.25	0.03	1.8	128.037	5.3961	1.27002	0.10343
155	0.25	0.06	1.8	133.977	4.36676	1.12887	0.09864
155	0.25	0.09	1.8	140.192	3.44977	0.99987	0.09356
155	0.4	0.03	1.8	113.37	37.3307	1.23082	0.38203
155	0.4	0.06	1.8	118.629	34.0763	1.25481	0.33751
155	0.4	0.09	1.8	124.133	31.0608	1.26214	0.29733
160	0.1	0.03	2	148.881	10.9906	1.2952	0.12809
160	0.1	0.06	2	156.428	10.6137	1.28869	0.09511
160	0.1	0.09	2	164.359	10.211	1.2697	0.06669

160	0.25	0.03	2	131.284	9.26208	4.02127	0.54735
160	0.25	0.06	2	137.939	7.52659	3.57677	0.52252
160	0.25	0.09	2	144.932	5.97884	3.1703	0.49611
160	0.4	0.03	2	115.766	62.3955	3.78456	2.04843
160	0.4	0.06	2	121.635	56.9734	3.87014	1.81069
160	0.4	0.09	2	127.801	51.9475	3.90233	1.59596

Table 7.3 Percentage errors obtained using the equation (7.17b)

(x, y, z, t)		% errors for n=3	% errors for n=10
	Exact Solution	in ADM	in ADM
(.1, .1, .1, .1)	0.099609	0.0029328	4.1017e-11
(.2, .3, .4, .5)	0.20001	1.2862	0.0007352
(.3, .2, .3, .4)	0.29929	0.56596	0.00010083
(.4, .5, .2, .3)	0.36525	0.27658	2.8212e-6
(.4, .4, .5, .7)	0.40429	4.36	0.014367

Table 7.4 Percentage errors obtained using the equation (7.17b)

x	y	z	t	Exact solution	% error for n=3	% error for n=5	% error for n=7
0.1	0.1	0.1	0.2	0.09866	0.04773	0.00137	2.01E-05
0.1	0.1	0.5	0.1	0.10348	4.10E-05	2.01E-05	0.000
0.1	0.2	0.3	0.3	0.10028	0.17804	0.01234	0.0004
0.1	0.2	0.6	0.4	0.11012	0.12036	0.06420	0.00015
0.1	0.3	0.3	0.4	0.09706	0.6741	0.04318	0.00476
0.1	0.3	0.6	0.2	0.1046	0.01422	0.00098	0.000
0.1	0.5	0.2	0.1	0.09699	0.00360	0.000	0.000
0.1	0.5	0.3	0.4	0.09187	0.76673	0.00603	0.00402
0.1	0.8	0.1	0.3	0.08316	0.09682	0.01642	0.00065
0.2	0.1	0.6	0.3	0.2218	0.02213	0.01313	0.00024
0.2	0.2	0.5	0.4	0.2132	0.26099	0.06712	0.00105

0.2	0.3	0.3	0.3	0.19608	0.21557	0.00831	0.00044
0.2	0.5	0.3	0.1	0.19584	0.00327	0.000	0.000
0.2	0.5	0.6	0.4	0.2025	0.45443	0.02485	0.00267
0.2	0.7	0.2	0.4	0.16821	0.62777	0.06713	0.00117
0.4	0.2	0.5	0.4	0.42628	0.26099	0.06712	0.00106
0.5	0.2	0.6	0.4	0.55071	0.12023	0.06432	0.00011
0.5	0.4	0.3	0.2	0.48681	0.04821	0.00031	2.10E-05
0.6	0.2	0.1	0.3	0.57130	0.28239	0.0095	0.00079
0.7	0.5	0.3	0.2	0.6707	0.0506	4.01E-05	2.01E-05
0.7	0.8	0.3	0.2	0.63789	0.03188	0.00101	1.00E-05
0.8	0.3	0.5	0.4	0.82940	0.39079	0.0533	0.00249
1	0.2	0.3	0.4	0.9977	0.56581	0.0649	0.00391
1	0.8	0.6	0.4	0.93371	0.47388	0.0279	0.00021

Table 7.5 represents the time elapsed in seconds for finding the solution of Three Factor model with the parameter values used in table 7.1

Method	Elapsed time in seconds for n=4	Elapsed time in seconds for n=10	Elapsed time in seconds for n=15	Elapsed time in seconds for n=25
ADM	2.220892	6.308238	35.613235	1373.568140
VIM	1.853144	6.246374	79.699470	> 1.5 hour

Table 7.6 represents the possible values of c_0 (convergence control parameter) obtained from (7.18) with $k = 1.314, \alpha = 0.249, \sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = 0.0081, \rho_1 = 0.915, \rho_2 = -0.0039, \rho_3 = -0.0293, \tilde{\lambda} = 0.353, a = 0.2, m^* = 0.07082$ and for various values of x, y, z, t

$x = 19.5, y = .10, z = .03, t = 0$	$x = 20, y = .10, z = .03, t = 0.2$
-2.91051662177178 + 1.95202089283503i	-2.55436347895358 + 1.57412780682948i
-2.91051662177178 -	-2.55436347895358 -

1.95202089283503i	1.57412780682948i
1.00932725604556 + 1.95216454138055i	0.668084284726043 + 1.57710895107762i
1.00932725604556 - 1.95216454138055i	0.668084284726043 - 1.57710895107762i
$x = 20.5, y = .10, z = .03, t = 0.4$	$x = 21, y = .10, z = .03, t = 0.6$
-2.43637862726395 + 1.35947648510095i	-2.46828086849421 + 1.16600585160155i
-2.43637862726395 - 1.35947648510095i	-2.46828086849421 - 1.16600585160155i
0.489740515113065 + 1.37797646087807i	0.368928028285936 + 1.24369098793811i
0.489740515113065 - 1.37797646087807i	0.368928028285936 - 1.24369098793811i
$x = 21.5, y = .10, z = .03, t = 0.8$	$x = 22, y = .10, z = .03, t = 1$
-2.67653908927327 + 0.831812460292432i	-4.17872063860209 + 0.00000000000000i
-2.67653908927327 - 0.831812460292432i	-2.23894244189587 + 0.00000000000000i
0.276628511433550 + 1.14093283674904i	0.201913389250992 + 1.05508107653545i
0.276628511433550 - 1.14093283674904i	0.201913389250992 - 1.05508107653545i
$x = 22.5, y = .10, z = .03, t = 1.2$	$x = 23, y = .10, z = .03, t = 1.4$
-7.49729674362432 + 0.00000000000000i	-24.1328970110900 + 0.00000000000000i
-1.83212518344587 + 0.00000000000000i	-1.60463896150629 + 0.00000000000000i
0.140434984267648 +	0.0904128074792712 +

0.978903652225325i	0.909145295348406i
0.140434984267648 - 0.978903652225325i	0.0904128074792712 - 0.909145295348406i
$x = 19.5, y = .15, z = .03, t = 0$	$x = 20, y = .15, z = .03, t = 0.2$
-2.29142051514755 + 1.39532452709706i	-1.90560663718182 + 1.07558448480444i
-2.29142051514755 - 1.39532452709706i	-1.90560663718182 - 1.07558448480444i
0.507783051146976 + 1.39514959724646i	0.272867192244906 + 1.07461420997982i
0.507783051146976 - 1.39514959724646i	0.272867192244906 - 1.07461420997982i
$x = 20.5, y = .15, z = .03, t = 0.4$	$x = 21, y = .15, z = .03, t = 0.6$
-1.68343482136240 + 0.899634757899800i	-1.52980794146682 + 0.780424918810946i
-1.68343482136240 - 0.899634757899800i	-1.52980794146682 - 0.780424918810946i
0.164943493874010 + 0.897537416427583i	0.102470615178616 + 0.777329967168531i
0.164943493874010 - 0.897537416427583i	0.102470615178616 - 0.777329967168531i
$x = 21.5, y = .15, z = .03, t = 0.8$	$x = 22, y = .15, z = .03, t = 1$
-1.41427308295932 + 0.691582922620829i	-1.32289335302662 + 0.621482560022472i
-1.41427308295932 - 0.691582922620829i	-1.32289335302662 - 0.621482560022472i
0.0623280119330451 + 0.687939520704833i	0.0349514499635184 + 0.617910290217611i
0.0623280119330451 -	0.0349514499635184 -

0.687939520704833i	0.617910290217611i
$x = 22.5, y = .15, z = .03, t = 1.2$	$x = 23, y = .15, z = .03, t = 1.4$
-1.24808424893323 + 0.563965800518490i	-1.18526285090381 + 0.515390644769821i
-1.24808424893323 - 0.563965800518490i	-1.18526285090381 - 0.515390644769821i
0.0155583958843659 + 0.561150355052783i	0.00146276615930262 + 0.514020243887365i
0.0155583958843659 - 0.561150355052783i	0.00146276615930262 - 0.514020243887365i
$x = 19.5, y = .15, z = .06, t = 0$	$x = 20, y = .15, z = .06, t = 0.2$
-2.33130227313091 + 1.44015191429721i	-1.93307813427265 + 1.10733259291355i
-2.33130227313091 - 1.44015191429721i	-1.93307813427265 - 1.10733259291355i
0.559516980513505 + 1.43988618489090i	0.313851258254845 + 1.10588324365860i
0.559516980513505 - 1.43988618489090i	0.313851258254845 - 1.10588324365860i
$x = 20.5, y = .15, z = .06, t = 0.4$	$x = 21, y = .15, z = .06, t = 0.6$
-1.70654972266153 + 0.925132390767941i	-1.55133740615974 + 0.802313482177420i
-1.70654972266153 - 0.925132390767941i	-1.55133740615974 - 0.802313482177420i
0.200173872827161 + 0.921981848914586i	0.133786969805682 + 0.797541708814776i
0.200173872827161 - 0.921981848914586i	0.133786969805682 - 0.797541708814776i
$x = 21.5, y = .15, z = .06, t = 0.8$	$x = 22, y = .15, z = .06, t = 1$

-1.43537554457298 + 0.711162753497807i	-1.34409457281070 + 0.639466584996773i
-1.43537554457298 - 0.711162753497807i	-1.34409457281070 - 0.639466584996773i
0.0906895242352878 + 0.705247803187147i	0.0609521024802013 + 0.633095649154331i
0.0906895242352878 - 0.705247803187147i	0.0609521024802013 - 0.633095649154331i
$x = 22.5, y = .15, z = .06, t = 1.2$	$x = 23, y = .15, z = .06, t = 1.4$
-1.26962878989814 + 0.580768389451764i	-1.20726040974519 + 0.531260159418163i
-1.26962878989814 - 0.580768389451764i	-1.20726040974519 - 0.531260159418163i
0.0396052627992929 + 0.574711820712472i	0.0238536655695816 + 0.526296617605541i
0.0396052627992929 - 0.574711820712472i	0.0238536655695816 - 0.526296617605541i

7.7.2 DISCUSSION

This study has undertaken parameters as were available against pre-executed study by Schwartz [111] for crude oil and copper, say, speed adjustment of spot commodity prices (k), volatility of spot price of commodity (σ_1), volatility of convenience yield (σ_2), volatility of risk free interest (σ_3), log run mean price of the convenience yield (α), market price of risk ($\tilde{\lambda}$), correlation between spot price and convenience yield of a commodity (ρ_1), correlation between convenience yield and risk free interest rate (ρ_2), correlation between risk free interest rate and spot price of commodity (ρ_3), speed adjustment of interest rate (a), risk adjusted mean short rate of interest rate (m^*) and were substituted in the three-factor model, and subsequently, crude oil and copper future prices were found.

Previous to the obtained prices, on substitution of parameters, the achieved polynomial (A 3.1.1) and the corresponding coefficients were tested successfully for convergence (A 3.3.1 & A 3.3.2), and thus approximate solution was derived. On comparison with exact solutions derived from analytical solution of the existing study, the obtained approximate solution values from both ADM and VIM methods are found to be exactly same. The attained errors (table 7.1) are found to be reduced considerably with increased number of iterations.

On the similar lines, present study as refereed above, commodity prices for copper were subjected to three-factor model, and this study resultant approximate solution (A 3.1.2) using ADM and VIM methods, and as a matter of successful application the achieved values are precisely matching with exact solutions, and the errors (tables 7.2 to 7.4), were significantly reduced with increased number of iterations.

Further, the study has been inspected to accessible time gap between the ADM and VIM methods, for both crude oil and copper prices. The observed delay for 10 iterations or n=10, is mostly similar with both the methods, but further increase in the number of iterations, the delay mostly in case of VIM found to be about 6 times. Detailed delay specific tables were shown in tables 7.5.

The study further estimated for HAM method on the above refereed study, and thus obtained polynomial in terms of five variables, that are, x, y, z, t, and c_0 (7.18). When the parameters of the refereed study, $x, y, z,$ and t were substituted in the above polynomial, 8-sets of polynomials were derived (A 3.2.1 to A 3.2.8). However, the study observed that the c_0 value happens to be not common among all the obtained 8-sets of polynomials (table 7.6), an impediment while testing validity of HAM.

7.8 CONCLUSION

The validation of ADM, VIM, HAM and HPM methods, were undertaken for deriving approximate solutions of three-factor commodity price model, in the form of polynomials which in turn will be of immense help while efficaciously predicting the future commodity prices at any short interval of time more accurately and with less degree of error. The obtained errors are to be reduced considerably with increased number of iterations. However, computations through VIM are of longer duration processing compared to ADM to obtain the approximate polynomial. Absence of convergence control parameter c_0 is one of the important limitations while solving the three-factor commodity price model using HPM and HAM.

CHAPTER 8

CONCLUSIONS

The study is a part of cross validation using ADM, VIM and HAM methods onto the existing linear PDE models, selected from commodity market and validation of analytical methods including, FIM, Tanh-Coth, and Sine-Cosine methods on the non-linear PDE model selected from securities market.

The validation methods, largely non-discretization methods were undertaken for deriving approximate solutions of one-factor, two-factor and three-factor commodity price models, in the form of polynomials which in turn will be of immense help while efficaciously predicting the future commodity prices at any short interval of time (which happens to be a serious drawback with discretization method), more accurately and with less degree of error. However, time consumed during validation process varies with respect to the methods undertaken.

For examining non-linear Black-Scholes model, selected from securities market, analytical methods that include, FIM, Tanh-Coth, and Sine-Cosine methods were utilized for validation purpose of the available literature. The said methods could not be as successful when compared to non-discretization methods largely due to non-availability of first integral polynomials, M and β.

CHAPTER 9

FUTURE SCOPE OF RESEARCH

Future research is possible with the collection of larger group of Black-Scholes equations, segregated based upon commodity and security future prices and their meticulous examination for inherent parameter variations. Extension of the study onto weather-based commodity and security price fluctuations, political, military, economic or policy-based forecasts should be of immense utility while safeguarding interests of the trading business houses and retail investors.

Validation of PDE models may also be subjected against Indian commodities, like Potato, Jaggery and in understanding fluctuations of INR against foreign currencies.

Greater scrutiny of PDE models will extend opportunities for Indian researchers to promote relevance and importance of PDE as it has been popular and successful in international research.

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APPENDIX

A-1 ONE FACTOR COMMODITY PRICE MODEL

A 1.1 The following solutions obtained using ADM and VIM

A 1.1.1 considering a set of parameter values $\boldsymbol{k = .301}$, $\boldsymbol{\mu = 3.093}$, $\boldsymbol{\sigma_1 = .334}$, $\boldsymbol{\tilde{\lambda} = -.242, n = 4}$ for crude oil

$$u_4 = ((90601 * t^4 * (28712603007819453001250701777567744000000000 * log(x)^4 - 2107505060773947850291801510473472409600000 * log(x)^3 + 361534631073644820561975505341552944742400000 * log(x)^2 + 2934448706673709895860670760602825276784640 0 * log(x) - 768567292228980000802981622458652034223332661))/ 760590360136937640898021923225600000000000000000000 - (90601 * t^3 * (33889587195962982400000 0 * log(x)^3 - 2373965583077206917120000 * log(x)^2 + 4677016557554414831600200 * log(x) - 195300812246784581865673 9))/6755399441055744000000000000000000 - (301 * t * (200 * log(x) - 667))/200000 - (241957554510902209 * t^2)/28823037615171174400 + (t^2 * ((301 * log(x))/1000 - 140567/ 200000) * ((301 * log(x))/1000 - 200767/200000))/2 + 1) * x$$

A 1.1.2 considering a set of parameter values $\boldsymbol{k = .369}$, $\boldsymbol{\mu = 4.854}$, $\boldsymbol{\sigma_1 = .233}$, $\boldsymbol{\tilde{\lambda} = -.339, n = 4}$ for copper

$$u\_4 = ((136161 * t^4 * (898982153008728872589544818475008000000000000000 * log(x)^4 - 132797643642449429058927560585128181760000000000 * log(x)^3 + 673230833669936448131787589025242042859520000000 * log(x)^2 - 128729952555755501799313550680205948904013824000 * log(x) + 5846938425446604124634492633710273111469777863 3))/ 15845632502852867518708790067200000000000000000000000000 -$$

$(369 * t * (1000 * log(x) - 5193))/1000000 - (136161 * t^3 * (1731071106770534400000000 * log(x)^3 - 21775143452066552217600000 * log(x)^2 + 87459009405707864531209300 * log(x) - 109761742637028128839942073 3))/ 281474976710656000000000000000 - (144350391268555083 * t^2)/2882303761517117440 0 + (t^2 * ((369 * log(x))/1000 - 1547217/1000000) * ((369 * log(x))/1000 - 1916217/1000000))/2 + 1) * x$

A 1.2 The following eight sets of polynomials in terms of convergence control parameter c_0 are obtained by considering a set of parameter values $k =.301$, $\mu = 3.093$, $\sigma_1 =.334$, $\tilde{\lambda} = -.242$ using HAM for n=4

A 1.2.1 polynomial for $x = 20$ and $t = 0.2$

$HP1 = (17255753471483671947 * c\_0)/14073748835532800000 + (7949769648372197 * c\_0 * ((5043449171693303712239190119104 3 * c\_0^2)/20480 - (24344677104302768505418277732913 * c\_0)/5120 + 22938284921427894998684139 52))/ 446014903970612462830714365452967230119608 32 - (12099187640492189240756664344999247462624081173452 27 * c\_0^2)/ 713623846352979940529142984724747568191373312000000 + (46345101890100084644 9 * c\_0^3)/35184372088832000000 0 - (14409944409241286883080647805171466243 * c\_0^4)/316912650057057350374175801344000000 00 + 20$

A 1.2.2 polynomial for $x = 20.5$ and $t = 0.4$

$HP2 = (32801065590548098573 9 * c\_0)/14073748835532800000 0 + (325940555583260077 * c\_0 * ((31506404606631097702732212668123 *$

c_0^2)/12800 − (5864272654220571355164811172767 ∗ c_0)/1280 +
2127029894094829401281658880))/
892029807941224925661428730905934460239216640 −
(2389879177721087578493866474239267912933052587755 6877 ∗
c_0^2)/713623846352979940529142984724747568191373312000000 0 +
(9492347251804004612811 ∗ c_0^3)/35184372088832000000000 −
(30575362034090275765223846715264 3295301 ∗
c_0^4)/31691265005705735037417580134400000000000 + 41/2

A 1.2.3 polynomial for $x = 21$ and $t = 0.6$

HP3 = (9308587482870285926 61 ∗ c_0)/28147497671065600000 0 +
(500835487847448411 ∗ c_0 ∗
((1259786819807841060797175521220 37 ∗ c_0^2)/51200 −
(4513832271733020134637767918449 ∗ c_0)/1024 +
1964250933178275864848105472))/
892029807941224925661428730905934460239216640 −
(7066384652223006103323523216265392336979138962968 4563 ∗
c_0^2)/
1427247692705959881058285969449495136382746624000000 0 +
(29147406029657452499697 ∗ c_0^3)/70368744177664000000000 −
(3892565041999662931575817230058456872423 ∗
c_0^4)/253530120045645880299340641075200000000000 + 21

A 1.2.4 polynomial for $x = 21.5$ and $t = 0.8$

HP4 = (11678264582468473903 3 ∗ c_0)/28147497671065600000 +
(341840094880004471 ∗ c_0 ∗ ((6295249073742332456829240859156 3 ∗
c_0^2)/25600 − (2167349847505238379019981485049 ∗ c_0)/512 +
1805302423172414700141936640))/
446014903970612462830714365452967230119608320 −
(5789480228619938405017249920056262488614976167192453 ∗ c_0^2)/

8920298079412249256614287309059344602392166400000000 +
(3974482611567798141359 * c_0^3)/70368744177664000000 −
(8602651005506593456462684264869158781 *
c_0^4)/396140812571321687967719751680000000 + 43/2

A 1.2.5 polynomial for $x = 22$ and $t = 1.0$

HP5 =
(3412947861887104013 * c_0)/703687441776640000 +
(87447466132094167 * c_0 * ((2515264427002163094694756772669 *
c_0^2)/1024 − (2076338745151716942347906437539 * c_0)/512 +
1650008227564573845849374720))/
89202980794122492566142873090593446023921664 −
(22700787421243735950238370469105767489807888666884377 * c_0^2)/
285449538541191976211657193889899027276549324800000 +
(2030002872999301519 * c_0^3)/281474976710656000 −
(2919537833605077055119271286870029781 *
c_0^4)/1014120480182583521197362564300800000 + 22

A 1.2.6 polynomial for $x = 22.5$ and $t = 1.2$

HP6 =

(7606119905455729041 * c_0)/1407374883553280000 +

(214643780506049319 * c_0 * ((62755533024443342540538991830737 *

c_0^2)/25600 − (4958180204745814297825170411281 * c_0)/1280 +

1498204086283919707577778176))/

178405961588244985132285746181186892047843328 −

(26610367162698463682431293953920244898843400193713689 *

c_0^2)/2854495385411919762116571938898990272765493248000000 +

(97062072996430355391 * c_0^3)/10995116277760000000 −

(46425985496306207610234042629102643 27 *
c_0^4)/12676506002282294014967032053760000000 + 45/2

A 1.2.7 polynomial for $x = 23$ and $t = 1.4$

HP7 = (102144856827352810037 * c_0)/17592186044416000000 +
(1279912913387923717 * c_0 * ((1000840018578460731763740349926 3 *
c_0^2)/4096 − (943796856265141397377070696722 9 * c_0)/2560 +
1349736571398595196674375680))/
8920298079412249256614287309059344602392166 40 −
(377537731784891244522783122656841755237441764746389 11 *
c_0^2)/356811923176489970264571492362373784095686656 0000000 +
(18447934068887906687251 * c_0^3)/17592186044416000000 00 −
(286932111183233626650301491024951625801 3 *
c_0^4)/6338253001141147007483516026880000000 00 + 23

A 1.2.8 polynomial for $x = 23.5$ and $t = 1.6$

HP8 = (212857995596236379489 * c_0)/35184372088832000000 +
(373639173473493259 * c_0 * ((311250595676077589993694456731 47 *
c_0^2)/12800 − (447199176036119984141618359997 1 * c_0)/1280 +
120446215513098760627001753 6))/
2230074519853062314153571827264836150598041 60 −
(208877537159978345477797080683708479353261340771701 51 *
c_0^2)/178405961588244985132285746181186892047843328 0000000 +
(107103986972685339015 07 * c_0^3)/87960930222080000000 0 −
(678050992915014814126919989730437637 *
c_0^4)/12379400392853802748991242240000000 0 + 47/2

A 1.3 Convergence values η_i from the section 5.4.2

A 1.3.1 considering a set of parameter values $k = .301$, $\mu = 3.093$, $\sigma_1 = .334$, $\tilde{\lambda} = -.242$, $x = 19.5 : 0.5 : 23.5$, $t = 0 : 0.2 : 1.6$, $n = 4, 7$ & 10 for crude oil

n=4	n=7	n=10
0.41221	0.41221	0.41221
0.174077	0.174077	0.174077
0.146277	0.146277	0.146277
	0.291929	0.291929
	0.130748	0.130748
	0.261371	0.261371
		0.13979
		0.21451
		0.156742

A 1.3.2 considering a set of parameter values $k = .694$, $\mu = 3.037$, $\sigma_1 = .326$, $\tilde{\lambda} = -.072$, $x = 13 : 0.5 : 18$, $t = 0 : 0.2 : 1.6$, $n = 4, 7$ & 10 for crude oil

n=4	n=7	n=10
0.890364	0.890364	0.890364
0.36839	0.36839	0.36839
0.369359	0.369359	0.369359
	0.604354	0.604354
	0.312121	0.312121
	0.556886	0.556886
		0.331587
		0.453809
		0.373594

A-2 TWO FACTOR COMMODITY PRICE MODEL

A 2.1 The following solutions obtained using ADM and VIM

A 2.1.1 considering a set of parameter values $\boldsymbol{\sigma_1 = 0.393, \sigma_2 = 0.527, \rho_1 = 0.766, r = 0.06, k = 1.876, \alpha = 0.106, \tilde{\lambda} = 0.198, n = 4}$ for crude oil

$$u_4 = ((t^4 * (19807040628566084398385987584000000000000 * y^4 + 2181943595642839857326200392253440000000000 * y^3 + 442677332595995590920580243426115584000000 * y^2 + 3583105825016249987421404455120561766400 0 * y - 378918390823937309501686136784110178449 97))/ 47536897508558602556126370201600000000000 0 - (t^3 * (17592186044416000000000 * y^3 + 95842229569978368000000 * y^2 + 47744962853935933844875 * y - 9648647866149896704441))/ 10555311626649600000000 0 + (t^2 * (1407374883553280000 00 * y^2 + 2471350295519559680 00 * y - 2194140030623540184 3))/ 2814749767106560000 00 - t * (y - 3/50) + 1) * x$$

A 2.1.2 considering a set of parameter values $\boldsymbol{\sigma_1 = 0.274, \sigma_2 = 0.280, \rho_1 = 0.818, r = 0.06, k = 1.156, \alpha = 0.248, \tilde{\lambda} = 0.256, n = 4}$ for copper

$$u_4 = ((t^4 * (316912650057057350374175801344000000000000 * y^4 + 21220471047820560181054811657994240000000000 * y^3 + 2529901718783822672560207979270373376000000 * y^2 - 2483137074020923287439335440455932313600 0 * y - 1038945084241776275990734897707765048224 29))/ 760590360136937640898021923225600000000000 0 - (t^3 * (703687441776640000000000 * y^3 + 231372430856159232000000 * y^2 + 60427126813237108953625 * y - 11949899169564909344282))/422212465065984000000000 - t * (y - 3/50) + (t^2 * (562949953421312000000 * y^2 +$$

$5832161517444479232000 * y - 5057821604713964008 1))/$
$1125899906842624000000 + 1) * x$

A 2.2 The following eight sets of polynomials in terms of convergence control parameter c_0 are obtained by considering the set of parameter values $\sigma_1 = 0.393, \sigma_2 = 0.527, \rho_1 = 0.766, r = 0.06, k = 1.876, \alpha = 0.106, \tilde{\lambda} = 0.198$ using HAM for n=4

A 2.2.1 polynomial for $x = 20, y = .10, t = 0.2$

$HP1 = -(628112726434687397 * c\_0\text{^}4)/3145728000000000000 +$

$(18587766165791057 * c\_0\text{^}3)/786432000000000000 +$

$(29893952840663 * c\_0\text{^}2)/419430400000000 - (16 * c\_0)/25 + 20$

A 2.2.2 polynomial for $x = 20.5, y = .10, t = 0.4$

$HP2 =$

$-(3219077722977773269 7 * c\_0\text{^}4)/98304000000000000000 +$

$(38104920639871673 * c\_0\text{^}3)/196608000000000000 +$

$(980521653173747 * c\_0\text{^}2)/3355443200000000 - (164 * c\_0)/125 +$

$41/2$

A 2.2.3 polynomial for $x = 21, y = .10, t = 0.6$

$HP3 =$

$-(35613991588846775229 * c\_0\text{^}4)/209715200000000000000 +$

$(1405235122133803551 * c\_0\text{^}3)/2097152000000000000 +$

$(9039931339016493 * c\_0\text{^}2)/13421772800000000 - (252 *$

$c\_0)/125 + 21$

A 2.2.4 polynomial for $x = 21.5, y = .10, t = 0.8$

$HP4 = -(5626843174310740713 * c\_0^4)/10240000000000000000 + (3197095780516061 * c\_0^3)/1966080000000000 + (1028351977718807 * c\_0^2)/838860800000000 - (344 * c\_0)/125 + 43/2$

A 2.2.5 polynomial for $x = 22, y = .10, t = 1.0$

$HP5 = -(1381847998156312261 * c\_0^4)/1006632960000000000 + (81786171129480643 * c\_0^3)/25165824000000000 + (526133569995669 * c\_0^2)/268435456000000 - (88 * c\_0)/25 + 22$

A 2.2.6 polynomial for $x = 22.5, y = .10, t = 1.2$

$HP6 = -(23848655081817039326 1 * c\_0^4)/81920000000000000000 + (1505609059429075473 * c\_0^3)/262144000000000000 + (2421410180093703 * c\_0^2)/838860800000000 - (108 * c\_0)/25 + 45/2$

A 2.2.7 polynomial for $x = 23, y = .10, t = 1.4$

$HP7 = -(115620896973009140887 3 * c\_0^4)/20971520000000000000 + (1955185163759008747 3 * c\_0^3)/2097152000000000000 + (53904775762283551 * c\_0^2)/134217728000000000 - (644 * c\_0)/125 + 23$

A 2.2.8 polynomial for $x = 23.5, y = .10, t = 1.6$

$HP8 = -(1845081133901894078 9 * c\_0\verb|^|4)/1920000000000000000 +$

$(7280208414934831 * c\_0\verb|^|3)/512000000000000 +$

$(1124012626808929 * c\_0\verb|^|2)/209715200000000 - (752 * c\_0)/125 +$

$47/2$

A 2.3 Convergence values η_i from the section 5.4.2

A 2.3.1 considering a set of parameter values $\sigma_1 = .374, \sigma_2 = .556, \rho_1 = .882, \tilde{\lambda} = .316, \alpha = .184, k = 1.829, r = 0.06, n = 4, 7$ & 10 for crude oil

n=4	n=7	n=10
0.993304	0.993304	0.993304
0.996712	0.996712	0.996712
0.992021	0.992021	0.992021
	0.905004	0.905004
	0.867625	0.867625
	0.842512	0.842512
		0.812944
		0.790463
		0.774609

A 2.3.2 considering a set of parameter values $\sigma_1 = .274, \sigma_2 = .280, \rho_1 = .818, \tilde{\lambda} = .256, \alpha = .248, k = 1.156, r = 0.06, n = 4, 7$ & 10 for copper

n=4	n=7	n=10
0.720052	0.720052	0.720052
0.8194	0.8194	0.8194
0.738257	0.738257	0.738257
	0.69069	0.69069
	0.659599	0.659599
	0.627669	0.627669
		0.602973
		0.58253
		0.564354

A-3 THREE FACTOR COMMODITY PRICE MODEL

A 3.1 The following solutions obtained using ADM and VIM

A 3.1.1 considering a set of parameter values $\boldsymbol{\sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = .0081, \rho_1 = 0.915, \rho_2 = -.0039, \rho_3 = -.0293, r = 0.06, k = 1.314, \alpha = 0.249, \tilde{\lambda} = 0.353, a = .2, m^* = 0.07082, n = 4}$ for crude oil

$$u_4 = ((y^4/24 - (y^3 * z)/6 + (657 * y^3)/2000 + (y^2 * z^2)/4 - (707 * y^2 * z)/1000 + (111670111421602706170 6313 * y^2)/2305843009213693952000000 - (y * z^3)/6 + (857 * y * z^2)/2000 - (11121155285880232118914 19 * y * z)/34587645138205409280000000 + (9191231358087154633775179 * y)/345876451382054092800000000 + z^4/24 - z^3/20 - (5279405382372575896502 9 * z^2)/69175290276410818560000000 + (1276295910593650082432933 * z)/27670116110564327424000000 - 7585021189897324533043556793318592716159584235 1/2658455991569831745807614120560689152000000000000000) * t^4 + ((14409657103144088 53 * y)/36893488147419103232 - (14409657103144088 53 * z)/36893488147419103232 - (((657 * y)/500 + 2325118415618834 73/9007199254740992000) * (y - z + 657/1000))/3 + ((z/5 - 3541/250000) * (y - z + 1/10))/3 - ((y - z) * ((657 * y)/1000 - z/10 + (y - z)^2/2 - 44499519715401682428343/1152921504606846976000000))/3 + 71904583791490514412 3/147573952589676412928 00) * t^3 + ((657 * y)/1000 - z/10 + (y - z)^2/2 - 44499519715401682428343/1152921504606846976000000) * t^2 + (z - y) * t + 1) * x$$

A 3.1.2 considering a set of parameter values $\boldsymbol{\sigma_1 = .266, \sigma_2 = 0.249, \sigma_3 = .0096, \rho_1 = 0.805, \rho_2 = .1243, \rho_3 = .0964, r = 0.06, k = 1.045, \alpha = 0.255, \tilde{\lambda} = 0.243, a = .2, m^* = 0.071152, n = 4}$ for copper

$$
\begin{aligned}
u_4 = {} & ((y\text{^}4/24 - (y\text{^}3 \ast z)/6 + (209 \ast y\text{^}3)/800 + (y\text{^}2 \ast z\text{^}2)/4 - \\
& (229 \ast y\text{^}2 \ast z)/400 + (261939232784139211396525 7 \ast y\text{^}2)/ \\
& 8646911284551352320000000 - (y \ast z\text{^}3)/6 + (289 \ast y \ast z\text{^}2)/800 - \\
& (9068985695337610099652 57 \ast y \ast z)/4323455642275676160000000 + \\
& (4684442840930330162975411 \ast \\
& y)/576460752303423488000000000 + z\text{^}4/24 - z\text{^}3/20 - \\
& (3383133502989767649874 3 \ast z\text{^}2)/8646911284551352320000000 + \\
& (2241678686028222739577 71 \ast z)/8646911284551352320000000 - \\
& 137569764088320164410750484395591850858452073317/ \\
& 124615124604835863084731911901282304000000000000000) \ast t\text{^}4 + \\
& ((163168224238515099 \ast y)/9223372036854775808 - \\
& (163168224238515099 \ast z)/9223372036854775808 - ((y - z) \ast \\
& ((209 \ast y)/400 - z/10 + (y - z)\text{^}2/2 - 449039888943322622995 81/ \\
& 1441151880758558720000000))/3 - (((209 \ast y)/200 - 939/40000) \ast \\
& (y - z + 209/400))/3 + ((z/5 - 4447/312500) \ast (y - z + \\
& 1/10))/3 + 422139938384214661/216172782113783808 00) \ast t\text{^}3 + \\
& ((209 \ast y)/400 - z/10 + (y - z)\text{^}2/2 - 449039888943322622995 81/ \\
& 1441151880758558720000000) \ast t\text{^}2 + (z - y) \ast t + 1) \ast x
\end{aligned}
$$

A 3.2 The following eight sets of polynomials in terms of convergence control parameter $\boldsymbol{c_0}$ are obtained by considering the set of parameter values $\boldsymbol{\sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = .0081, \rho_1 = 0.915, \rho_2 = -.0039, \rho_3 =}$

$-.0293, r = 0.06, k = 1.314, \alpha = 0.249, \tilde{\lambda} = 0.353, a = .2, m^* = 0.07082$,
using HAM for n=4

A 3.2.1 polynomial for $x = 20, y = .10, t = 0.2$

$HP1 =$

$(5026612180361843299610834822699 *$

$c\_0\^4)/4503599627370496000000000000000 +$

$(1781218307889615171497872O1 *$

$c\_0\^3)/45035996273704960000000000 + (2939146806851461098643 *$

$c\_0\^2)/562949953421312000000 + (76 * c\_0)/25 + 20$

A 3.2.2 polynomial for $x = 20.5, y = .10, t = 0.4$

$HP2 =$

$(35192283089109065269018886193159 *$

$c\_0\^4)/1125899906842624000000000000000 +$

$(2279382355254207164996763241 *$

$c\_0\^3)/225179981368524800000000000 +$

$(67880457435085659284363 * c\_0\^2)/5629499534213120000000 +$

$(779 * c\_0)/125 + 41/2$

A 3.2.3 polynomial for $x = 21, y = .10, t = 0.6$

$HP3 =$

$(563926752845141960980005228360999 *$

$c\_0\^4)/9007199254740992000000000000000 +$

$(16989854830450164679293950967 *$

$c\_0\^3)/900719925474099200000000000 +$

$(23205022125717712492352 7 * c\_0\text{^}2)/1125899906842624000000 +$

$(1197 * c\_0)/125 + 21$

A 3.2.4 polynomial for $x = 21.5, y = .10, t = 0.8$

$HP4 =$

$(758403398503384555372626495 7307 *$

$c\_0\text{^}4)/703687441776640000000000000000 +$

$(860070514361844783618385643 *$

$c\_0\text{^}3)/2814749767106560000000000 +$

$(4359589254447468452164 9 * c\_0\text{^}2)/1407374883553280000000 +$

$(1634 * c\_0)/125 + 43/2$

A 3.2.5 polynomial for $x = 22, y = .10, t = 1.0$

$HP5 =$

$(1221852187176337800730094389689 *$

$c\_0\text{^}4)/72057594037927936000000000000 +$

$(164303380498662613883576011 *$

$c\_0\text{^}3)/36028797018963968000000000 + (97405591444756641490 73 *$

$c\_0\text{^}2)/2251799813685248000000 + (418 * c\_0)/25 + 22$

A 3.2.6 polynomial for $x = 22.5, y = .10, t = 1.2$

$HP6 =$

$(5653001751026806563981044018 5071 *$

$c\_0\text{^}4)/225179981368524800000000000000000 +$

$(290149994213673352162932984 3 *$

$c\_0\text{^}3)/4503599627370496000000000 +$

$(6479489569188851539 0083 * c\_0^2)/1125899906842624000000 +$
$(513 * c\_0)/25 + 45/2$

A 3.2.7 polynomial for $x = 23, y = .10, t = 1.4$

$HP7 =$

$(320239874658751545962508554385 6877 *$

$c\_0^4)/90071992547409920000000000000000 +$

$(787823628315731534753936 60689 *$

$c\_0^3)/900719925474099200000000000 +$

$(832646424498854150650661 * c\_0^2)/1125899906842624 0000000 +$

$(3059 * c\_0)/125 + 23$

A 2.2.8 polynomial for $x = 23.5, y = .10, t = 1.6$

$HP8 =$

$(107015714205705424137940929 1239 *$

$c\_0^4)/2199023255552000000000000000 +$

$(405268943397385144767166447 *$

$c\_0^3)/35184372088832000000000000 + (325698951569200322762 21 *$

$c\_0^2)/35184372088832000000 + (3572 * c\_0)/125 + 47/2$

A 3.3 Convergence values η_i from the section 5.4.2

A 3.3.1 considering a set of parameter values $\sigma_1 = 0.344, \sigma_2 = 0.372, \sigma_3 = .0081, \rho_1 = 0.915, \rho_2 = -.0039, \rho_3 = -.0293, r = 0.06, k = 1.314, \alpha = 0.249, \tilde{\lambda} = 0.353, a = .2, m^* = 0.07082, n = 4, 7$ & 10 for crude oil

n=4	n=7	n=10
0.985682	0.985682	0.985682
0.917339	0.917339	0.917339
0.876781	0.876781	0.876781

	0.823456	0.823456
	0.774714	0.774714
	0.735843	0.735843
		0.703755
		0.676473
		0.653333

A 3.3.2 considering a set of parameter values $\sigma_1 =.266, \sigma_2 = 0.249, \sigma_3 = .0096, \rho_1 = 0.805, \rho_2 =.1243, \rho_3 =.0964, r = 0.06, k = 1.045, \alpha = 0.255, \tilde{\lambda} = 0.243, a =.2, m^* = 0.071152, n = 4, 7$ & 10 for copper

n=4	n=7	n=10
0.944451	0.944451	0.944451
0.837053	0.837053	0.837053
0.793577	0.793577	0.793577
	0.739345	0.739345
	0.692874	0.692874
	0.65535	0.65535
		0.624263
		0.598043
		0.5757

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